THE UNIVERSITY of EDINBURGH
informatics

lfcs | Laboratory for Foundations of Computer Science

GRAPH REWRITING AND COMBINATORIAL HOPF ALGEBRA

Nicolas Behr

based on

- Nicolas Behr, Vincent Danos, and Ilias Garnier: ***Stochastic mechanics of graph rewriting,*** accepted for publication in **LiCS 2016**
- Nicolas Behr, Vincent Danos, Ilias Garnier, and Tobias Heindel (2016): ***The algebras of graph rewriting,*** in preparation

Categories, Logic & Physics, Scotland, University of Edinburgh, April 14th 2016

MOTIVATION

- How do **graph transformation systems (GTSs)** compare to other types of stochastic systems?
- Why does the conventional GTS framework seem to have so many specialized conventions?
- Is it possible to re-use ideas from statistical physics and combinatorics to tackle GTS computations?

WHY GRAPH TRANSFORMATION SYSTEMS?

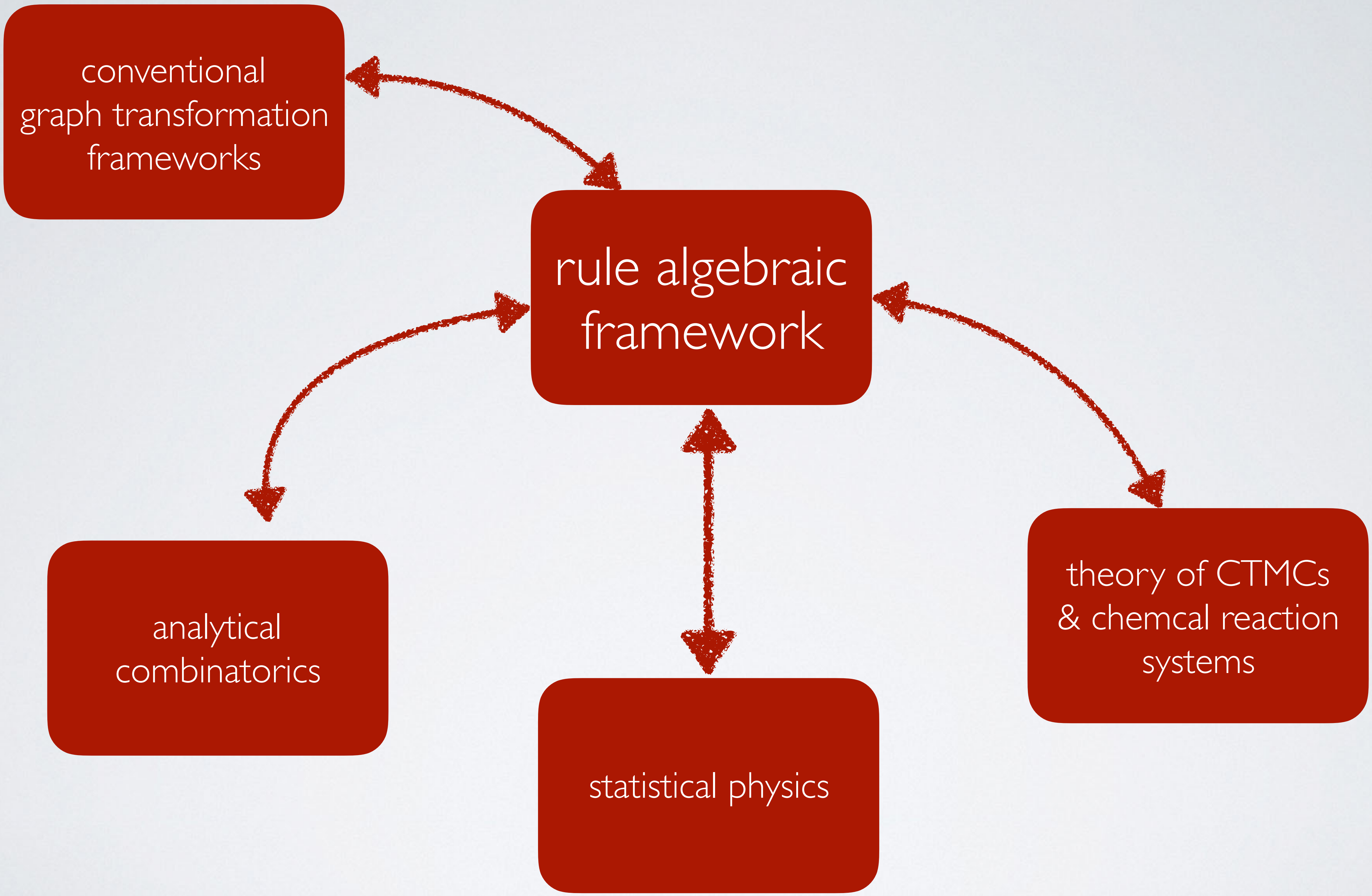
- allow to encode a variety of probabilistic systems (chemical reactions, **biochemical reactions**, network dynamics, lattice models,…)
- beyond “discrete” particles!
- very flexible modelling formalism

conventional
graph transformation
frameworks

analytical
combinatorics

statistical physics

theory of CTMCs
& chemcal reaction
systems


conventional graph transformation frameworks
rule algebraic framework
analytical combinatorics
statistical physics
theory of CTMCs & chemcal reaction systems

SURPRISING RESULTS

- ***entirely new formulation of graph rewriting syntax*** defined in terms of **rule diagrams** and their compositions rather than in terms of category theoretical concepts!
- discovery of ***new types of graph rewriting***
- in the new framework, we can study the **combinatorics** of rewriting systems
- the rule algebra (of type DPO) is a generalization of and subsumes the **Heisenberg-Weyl algebra**

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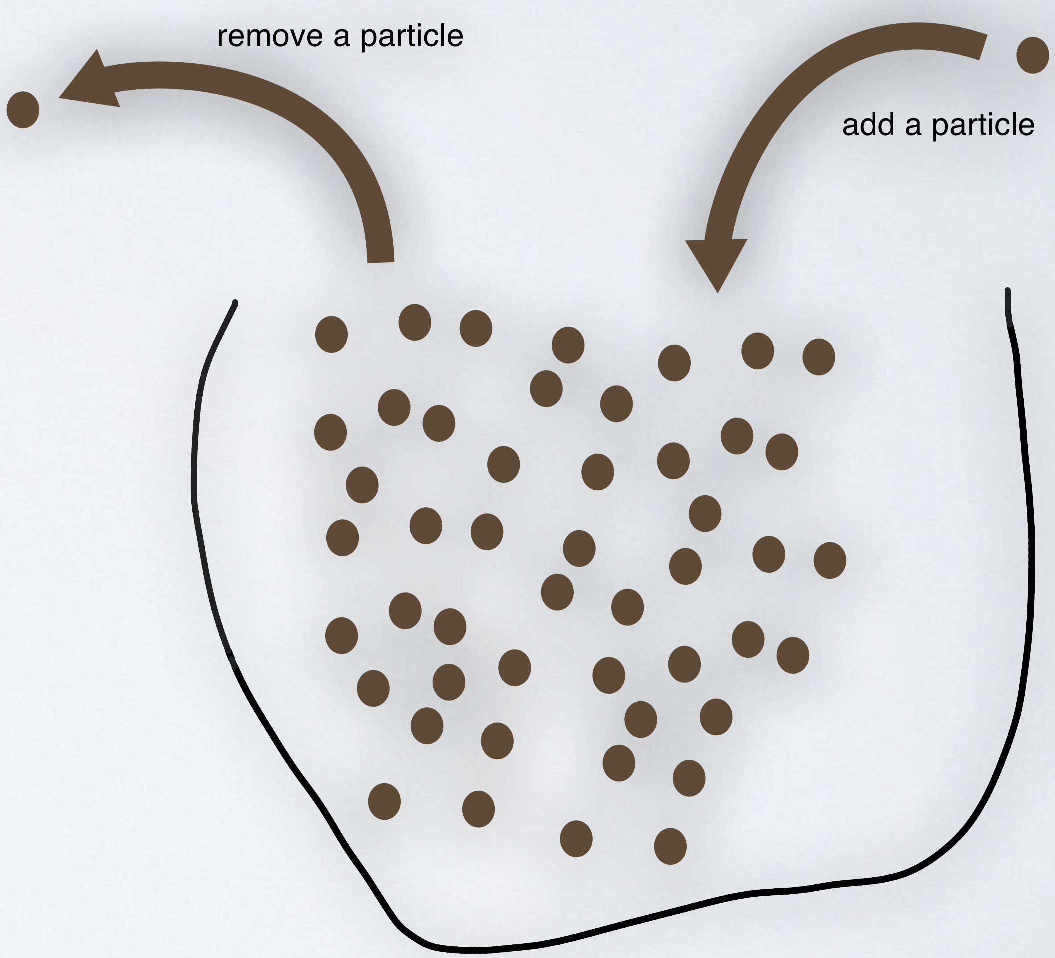
Focus of today's talk:
COMBINATORICS
of
DISCRETE GTS's…

STOCHASTIC MECHANICS OF CHEMICAL REACTION SYSTEMS

THE HEISENBERG-WEYL ALGEBRA

- pool of n ***indistinguishable particles***
- ***basic operations***:
  - pick a particle at random and ***remove it***
  - ***add a particle***
- ***basic combinatorics***:
  - n ***possibilities*** to remove a particle
  - 1 ***possibility*** to add a particle



THE HEISENBERG-WEYL ALGEBRA

- from the theory of ***bosonic Fock space****: introduce ***number vectors***

$$|n\rangle \quad \widehat{=} \quad \text{state of exactly } n \text{ particles}$$

- encode the basic operations in terms of the ***generators*** of the ***Heisenberg-Weyl algebra***:

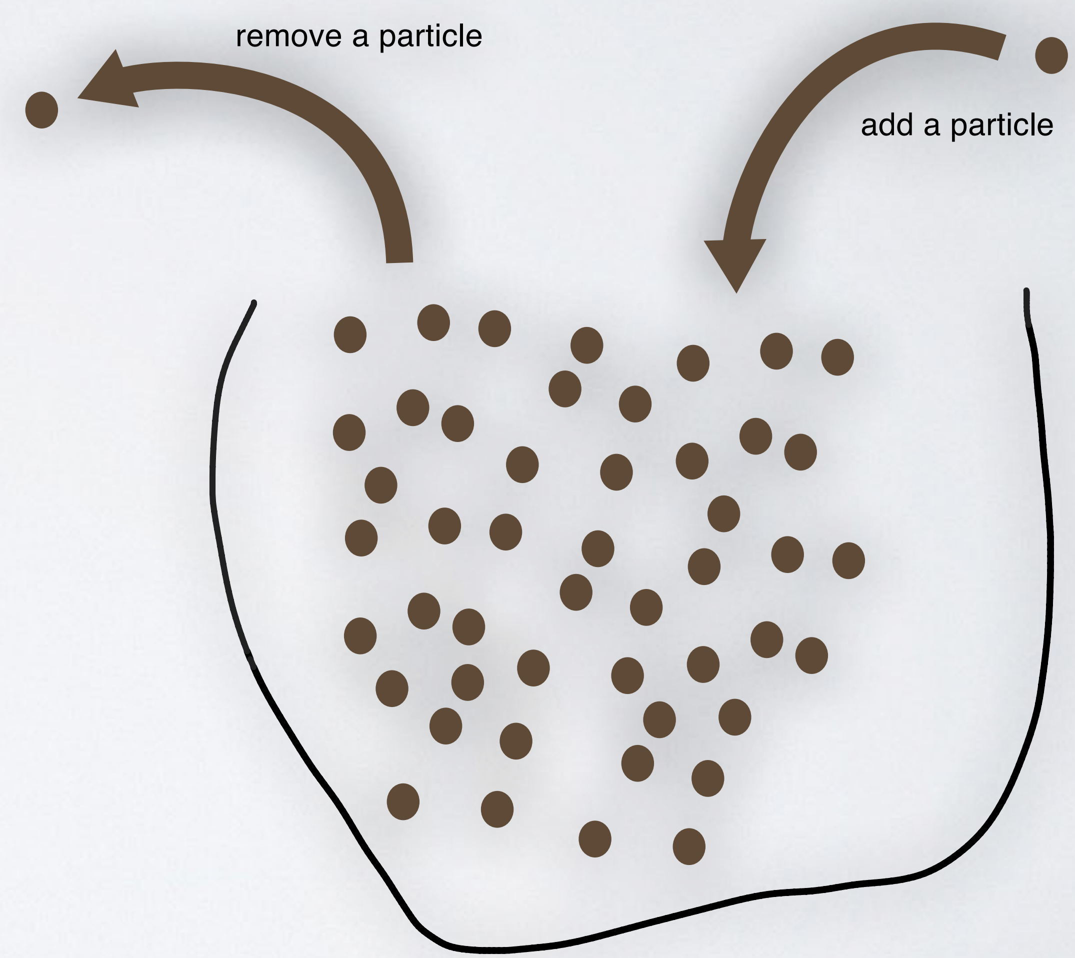
$$a|n\rangle := \begin{cases} n|n-1\rangle & \text{if } n > 0 \\ 0|0\rangle & \text{else.} \end{cases}$$

$$a^\dagger|n\rangle := |n+1\rangle \qquad (n \geq 0)$$

- ***canonical commutation relation:***

$$\begin{aligned}(aa^\dagger - a^\dagger a)|n\rangle &= a|n+1\rangle - na^\dagger|n-1\rangle \\ &= (n+1)|n\rangle - n|n\rangle = |n\rangle\end{aligned}$$

$$\Leftrightarrow \quad [a, a^\dagger] \equiv aa^\dagger - a^\dagger a = 1$$



$*$ Pawel Blasiak et al. (2007): ***Combinatorics and Boson normal ordering: A gentle introduction,*** American Journal of Physics 75.7, pp. 639–646

MULTI-TYPE CASE

- multi-type Heisenberg-Weyl algebra:

$$[a_i, a_j^\dagger] = \delta_{ij}$$

- multi-type bosonic Fock space:

$$|\vec{n}\rangle \equiv |n_1, n_2, \ldots\rangle \qquad (\sum_i n_i < \infty\,,\ n_i \in \mathbb{Z}_{\geqslant 0})$$

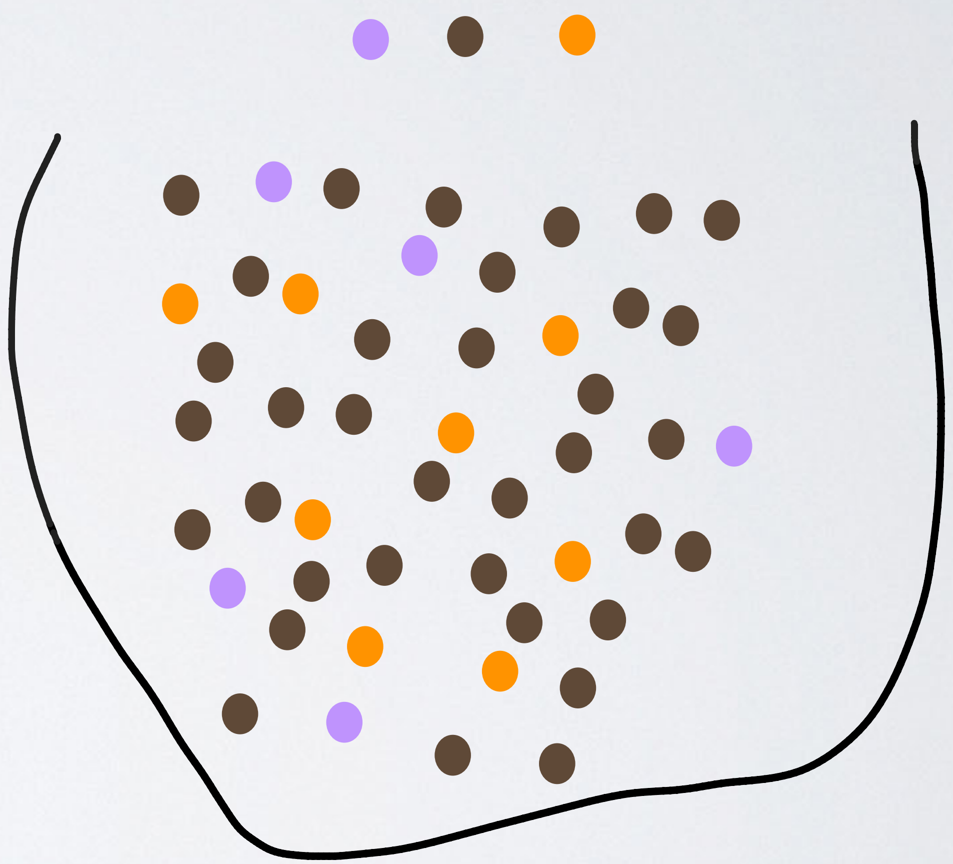
- action of the annihilators a_i and the creators $a_j^\dagger$:

$$a_i|\vec{n}\rangle := n_i|\vec{n} - \vec{\delta}_i\rangle\,, \quad a_i^\dagger|\vec{n}\rangle := |\vec{n} + \vec{\delta}_i\rangle$$

- interpretation: the individual types (visualized as colors) are acted upon independently from one another

example for a reaction:

$$A + B \to C$$



CHEMICAL REACTION SYSTEMS

$$\sum_{\tau \in T} \left(\sum_i s_i(\tau) X_i \xrightarrow{r_\tau} \sum_j t_j(\tau) X_j \right)$$

with: T – set of reactions; $r_\tau \in \mathbb{R}_{>0}$ – reaction rates; $s_i, t_j \in \mathbb{Z}_{\geq 0}$

- As presented e.g. in ***Baez and Bianmonte****, there exists a precise link between multi-type HW algebra and chemical reaction systems:
  - ***state of the system***:

$$|\Psi(t)\rangle = \sum_{\vec{n}} p_{\vec{n}}(t)|\vec{n}\rangle\,, \quad \sum_{\vec{n}} p_{\vec{n}} = 1\,, \quad p_{\vec{n}}(t) \in \mathbb{R}_{\geq 0}$$

  - ***Master equation***:

$$\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle\,,$$

with ***evolution operator***

$$H := \sum_{\tau \in T} r_\tau \left(\vec{a}^{\dagger\, \vec{t}(\tau)} - \vec{a}^{\dagger\, \vec{s}(\tau)} \right) \vec{a}^{\vec{s}(\tau)}$$

where $\vec{a}^{\dagger\, \vec{t}(\tau)} := \left(\prod_i a_i^{\dagger\, t_i(\tau)} \right)$ etc.

* John C Baez and Jacob Biamonte (2012): ***A course on quantum techniques for stochastic mechanics,*** arXiv preprint arXiv:1209.3632

CHEMICAL REACTION SYSTEMS

- ***Master equation***:

$$\frac{d}{dt}|\Psi(t)\rangle = H|\Psi(t)\rangle$$

- ***formal solution:***

$$|\Psi(t)\rangle = e^{tH}|\Psi(0)\rangle$$

- this is ***not*** always meaningful, but ***analytically solvable cases are known!***
- ***example: birth-death reaction system***

$$X \xrightarrow{1} \varnothing\,, \quad \varnothing \xrightarrow{\lambda} X$$

- ***analytic solution****: for $|\Psi(0)\rangle = |N_0\rangle$, and with $\nu(t) := (1 - e^{-t})$,

$$|\Psi(t)\rangle = e^{-\nu(t)\lambda} \sum_{k=0}^{min(N_0,n)} \binom{N_0}{k} e^{-kt}\nu(t)^{N_0+n-2k} \frac{\lambda^{n-k}}{(n-k)!}|n\rangle$$

* Tobias Jahnke and Wilhelm Huisinga (2007): ***Solving the chemical master equation for monomolecular reaction systems analytically,*** Journal of mathematical biology 54.1, pp. 1–26

GRAPH REWRITING

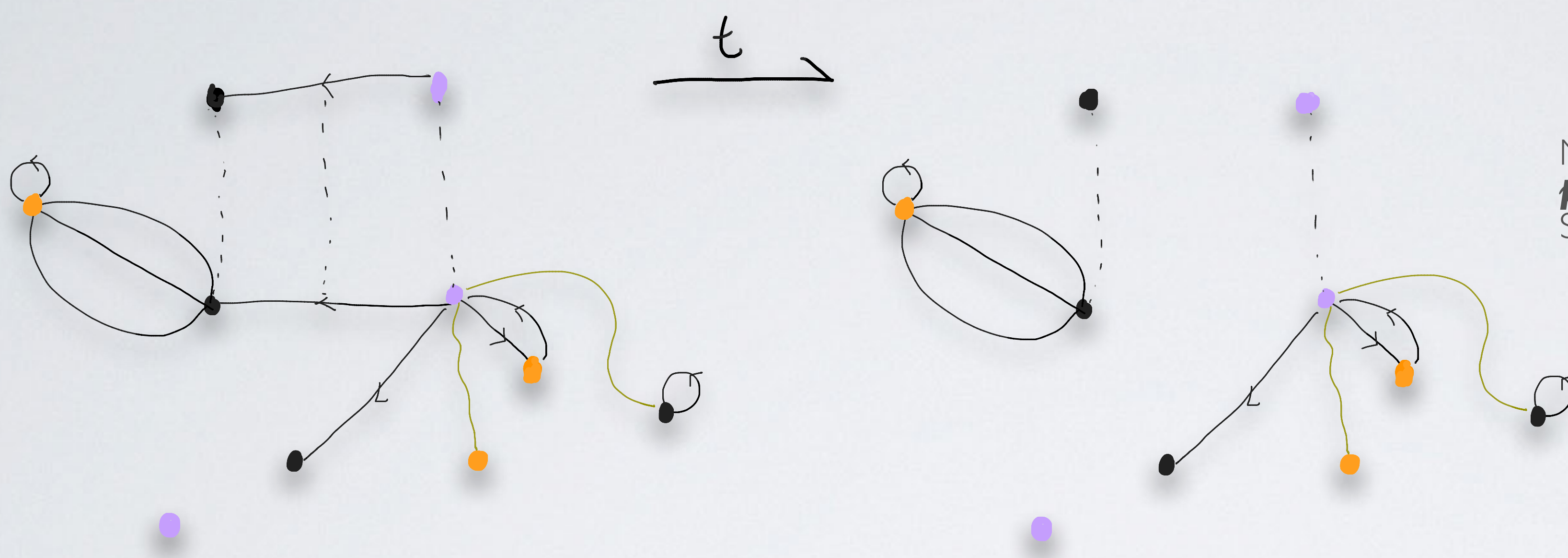
GRAPHS AND REWRITING RULES

- ***graphs*** = very generic graphs (multi-colored, multi edged, with self loops, directed or undirected edges?)
- ***injective partial graph homomorphisms*** = one-to-one mappings of vertices and edges (s.th. endpoint vertices of all edges are included)
- ***graph rewriting rule***:

$$t : L \to R$$

  with t – an injective partial homomorphism from finite graph L to finite graph R
- ***where it begins:*** a tentative depiction of graph rewriting rules…

ACTION OF REWRITING RULES ON GRAPHS



Michael Löwe (1993): ***Algebraic approach to single-pushout graph transformation,*** Theoretical Computer Science 109.1, pp. 181–224

- ***basic idea:*** to apply a rule $t : L \rightharpoonup R$ to a graph G,
  1. find an occurrence of L in G (aka a ***match***),
  2. then replace the instance of L with R as described in detail by the partial injection t
- a first hint at a notion of ***representation***: in general, there are more than one possibilities to act on a given graph, so which one to choose? ***answer: all of them***
- ***hint from HW algebra***: the outcome of a rewriting operation must be some sort of linear combination of ***"graph states"***

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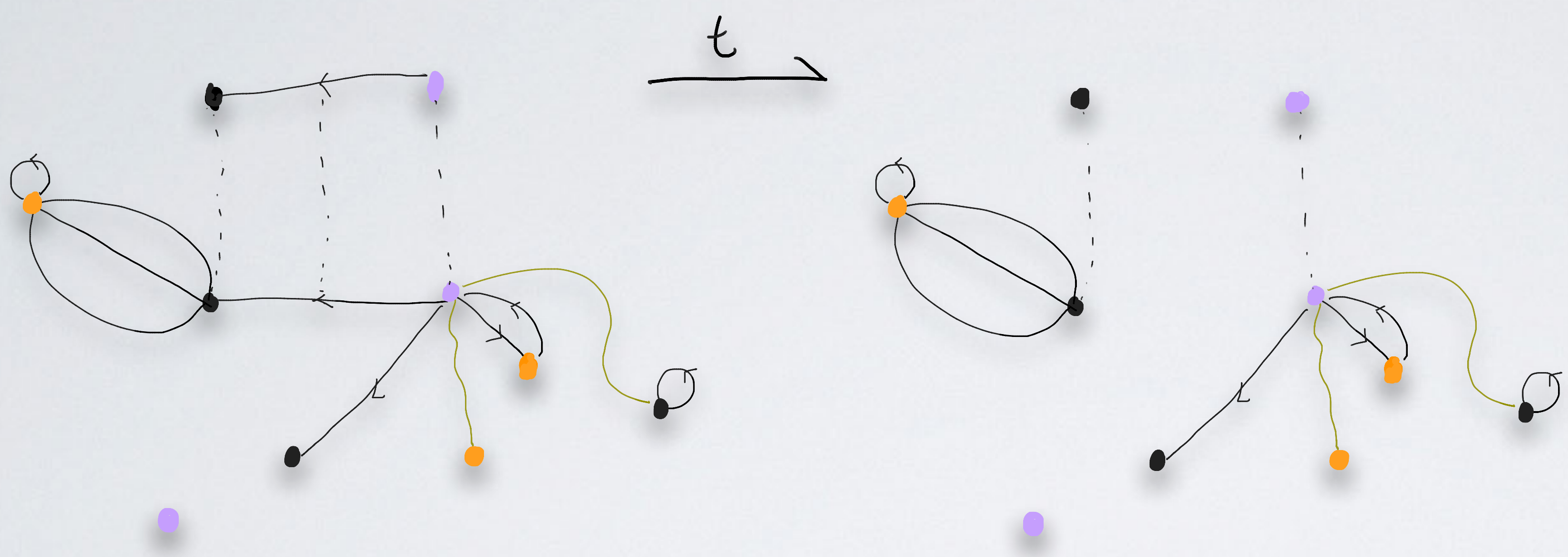


- there exist a ***number of different variants*** of graph rewriting:
  - ***double pushout (DPO) rewriting*** (deleting/creating a vertex is only possible if no edges remain after the operation)
  - ***single pushout (SPO) rewriting*** (deleting a vertex automatically deletes all incident edges)
  - ***our formalism: two new variants!***

Hartmut Ehrig et al. (2000): ***Double-pullback graph transitions: A rule-based framework with incomplete information,*** *Theory and Application of Graph Transformations.* Springer, pp. 85–102

Michael Löwe (1993): ***Algebraic approach to single-pushout graph transformation,*** Theoretical Computer Science 109.1, pp. 181–224

Nicolas Behr, Vincent Danos, Ilias Garnier, and Tobias Heindel (2016): ***Combinatorial physics of graph rewriting,*** in preparation

TOWARDS A NOTION OF ALGEBRA

- ***basic idea***:

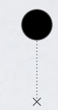
  composition of rules via ***compositions of diagrams***

- ***problem: HOW?!***
- ***Ansatz***: study the case of the Heisenberg-Weyl algebra in the formulation of ***combinatorial Hopf algebras***

RULE DIAGRAM ALGEBRA

DIAGRAMS FOR THE HW ALGEBRA

- ***Ansatz:*** Let us tentatively identify the two generators of the Heisenberg-Weyl (HW) algebra with the following diagrams:

$$d_a := \quad$$  $$, \quad d_{a^\dagger} := \quad$$  $$,$$

  depicting the actions of ***deleting a particle*** (d_a) and ***creating a particle*** ($d_{a^\dagger}$).

- Define a notion of ***composition along a match***: let $d_1, d_2 \in \{d_a, d_{a^\dagger}\}$, and let m_{12} denote a ***1:1 pairing*** of ***output vertices of*** d_2 with ***input vertices of*** d_1, including the possibility $m_{12} = \varnothing$. Then

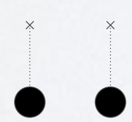
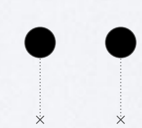
$$d_1 \overset{m_{12}}{\blacktriangleleft} d_2$$

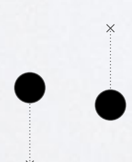
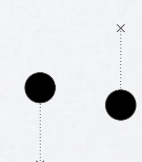
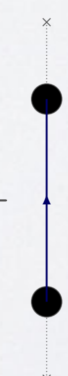
  denotes the diagram resulting from drawing d_1, d_2 and a linking arrow for each part of m_{12}.

- ***Composition of diagrams:*** Let $\mathcal{M}_{d_1}(d_2)$ denote the ***set of matches*** of d_2 into d_1. Then the ***composition*** $*_{\mathcal{D}}$ on the set of diagrams is defined as

$$d_1 *_{\mathcal{D}} d_2 := \sum_{m_{12} \in \mathcal{M}_{d_1}(d_2)} d_1 \overset{m_{12}}{\blacktriangleleft} d_2 \,.$$

  – ***Examples:***

$$d_a *_{\mathcal{D}} d_a = $$  $$\equiv d_a \uplus d_a \,, \quad d_{a^\dagger} *_{\mathcal{D}} d_{a^\dagger} = $$  $$\equiv d_{a^\dagger} \uplus d_{a^\dagger}$$

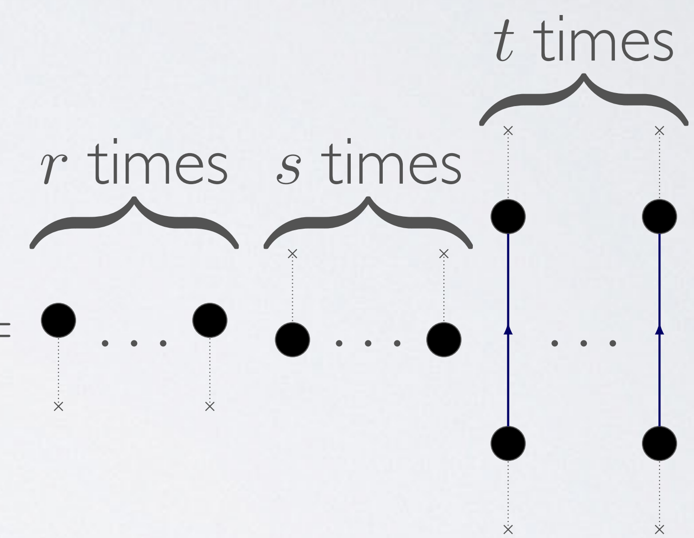
$$d_{a^\dagger} *_{\mathcal{D}} d_a = $$  $$\equiv d_{a^\dagger} \uplus d_a \,, \quad d_a *_{\mathcal{D}} d_{a^\dagger} = $$  $$+$$  $$\equiv d_{a^\dagger} \uplus d_a + d_e$$

- More precisely: we define diagrams ***up to isomorphism***, such that in particular

$$\forall d_1, d_2 \in \mathcal{D} : d_1 \uplus d_2 = d_2 \uplus d_1 \,.$$

- For the special case of the HW-rule diagrams, it turns out that any possible rule diagram has the form

$$\forall d \in \mathcal{D}_{HW} : \exists r, s, t \in \mathbb{Z}^3_{\geq 0} : d = d(r,s,t) =$$



$\Rightarrow$ ***purely combinatorial composition law:***

$$d(r_1, s_1, t_1) *_{\mathcal{D}} d(r_2, s_2, t_2) = \sum_{n=0}^{min(s_1, r_2)} \frac{s_1! r_2!}{(s_1 - n)! n! (r_2 - n)!} d(r_1 + r_2 - n, s_1 + s_2 - n, t_1 + t_2 + n) \,.$$

COMBINATORIAL HOPF ALGEBRA STRUCTURE

- ***Notation:*** from hereon, $\mathcal{D}_{HW}$ is simply denoted as $\mathcal{D}$!
- ***Algebra structure:*** Endow the space of isomorphism classes $\mathfrak{d} \in \mathcal{D}$ of rule diagrams with a ***vector space structure*** (over a field $\mathbb{K}$):

$$\mathcal{D} \equiv (\mathcal{D}, +, \cdot)\,, \qquad \forall d \in \mathcal{D} : d = \sum_{\mathfrak{d} \in \mathcal{D}} \kappa_{\mathfrak{d}}(d)\mathfrak{d}$$

$$(\alpha \cdot d_1 + \beta \cdot d_2) := \sum_{\mathfrak{d} \in \mathcal{D}} (\alpha \kappa_{\mathfrak{d}}(d_1) + \beta \kappa_{\mathfrak{d}}(d_2))\, \mathfrak{d}$$

  i.e. we have so far that $\mathcal{D} \equiv span_{\mathbb{K}}(\{\mathfrak{d} \in \mathcal{D})$.
- ***Binary composition law***:

$$\mu : \mathcal{D} \otimes \mathcal{D} \to \mathcal{D}_{HW} : d_1 \otimes d_2 \mapsto d_1 *_{\mathcal{D}} d_2$$

- ***Unit map***:

$$\eta : \mathbb{K} \to \mathcal{D} : k \mapsto k \cdot d_{\varnothing}$$

$\Rightarrow \mathcal{D} \equiv (\mathcal{D}, +, \cdot, \mu, \eta)$ is an ***associative, unital (noncommutative) algebra***.

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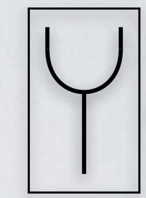
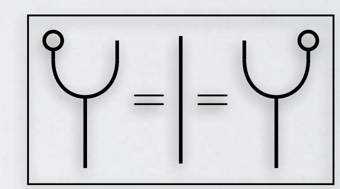
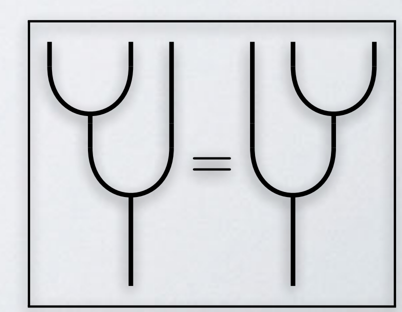
$\mu : A \otimes A \to A \quad \widehat{=}$ ,

$\eta : \mathbb{K} \to A \quad \widehat{=}$ 

$$\begin{tikzcd} & A \otimes A \arrow[dd, "\mu"] & \\ \mathbb{K} \otimes A \arrow[ur, "\eta\otimes Id_A"] \arrow[dr, "\cdot_L"'] & & A \otimes \mathbb{K} \arrow[ul, "Id_A\otimes\eta"'] \arrow[dl, "\cdot_R"] \\ & A & \end{tikzcd} \quad \widehat{=}$$



$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "Id_A\otimes\mu"] \arrow[d, "\mu\otimes Id_A"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \quad \widehat{=}$$



Coalgebra structure: via ***decompositions*** of rule diagrams!

- ***Coproduct:*** for a basis vector of $\mathcal{D}$ (i.e. isomorphism class $\mathfrak{d} \in \mathcal{D}$), define

$$\Delta(\mathfrak{d}) := \sum_{\text{ways to delete superposition factors from } \mathfrak{d}} (\mathfrak{d}_{\text{some factors deleted}}) \otimes (\mathfrak{d}_{\text{deleted factors}}) \,,$$

whence more concretely

$$\Delta(d(r,s,t)) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{\ell=0}^{t}\binom{r}{m}\binom{s}{n}\binom{t}{\ell} d(m,n,\ell) \otimes d(r-m, s-n, t-\ell)\,.$$

- ***counit map:***

$$\varepsilon : \mathcal{D} \to \mathbb{K} : \mathfrak{d} \mapsto \begin{cases} 1_{\mathbb{K}} & \text{if } \mathfrak{d} = d_{\varnothing} \equiv d(0,0,0) \\ 0_{\mathbb{K}} & \text{else.} \end{cases}$$

$\Rightarrow \mathcal{D} \equiv (\mathcal{D}, +, \cdot, \Delta, \varepsilon)$ is a ***coassociative, cocommutative and counital coalgebra***.

Pawel Blasiak, Gerard HE Duchamp, et al. (2010): ***Combinatorial algebra for second-quantized Quantum Theory,*** Advances in Theoretical and Mathematical Physics 14.4, pp. 1209–1243

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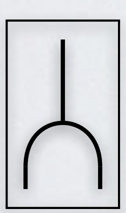
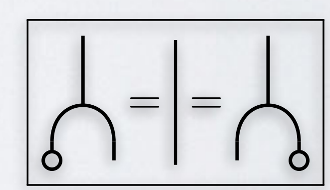


$\widehat{=} \quad \Delta : C \to C \otimes C$

$\widehat{=} \quad \varepsilon : C \to \mathbb{K}$

$$\begin{tikzcd}
& C \arrow[dl, "1_{\mathbb{K}}\otimes Id_C"'] \arrow[dd, "\Delta"] \arrow[dr, "Id_C\otimes 1_{\mathbb{K}}"] & \\
\mathbb{K}\otimes C & & C\otimes \mathbb{K} \\
& C\otimes C \arrow[ul, "\varepsilon\otimes Id_C"] \arrow[ur, "Id_C\otimes\varepsilon"'] &
\end{tikzcd}$$

$\widehat{=}$



$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C\otimes C \arrow[d, "Id_C\otimes\Delta"] \\
C\otimes C \arrow[r, "\Delta\otimes Id_C"'] & C\otimes C\otimes C
\end{tikzcd}$$

$\widehat{=}$

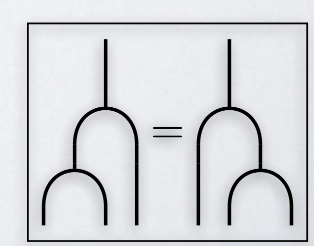
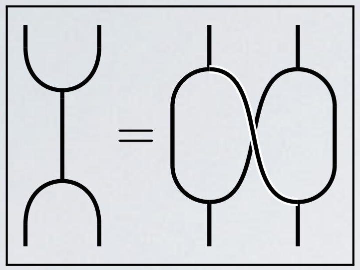
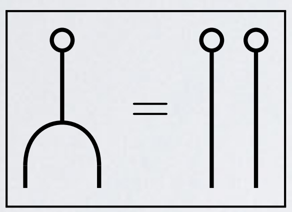
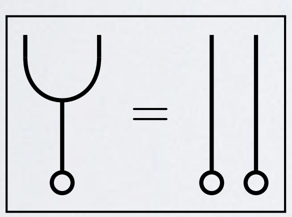
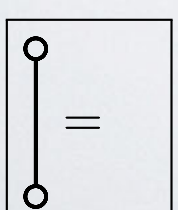


Table 1: Bialgebra compatibility conditions, expressed in each of the three equivalent forms: via string diagrams, from the viewpoint of the algebra structure, and from the viewpoint of the coalgebra structure.

string diagrams	algebra	coalgebra
	$\begin{tikzcd} A\otimes A \arrow[r,"\Delta_A\otimes\Delta_A"] \arrow[dd,"\mu_A"] & A\otimes A\otimes A\otimes A \arrow[d,"Id_A\otimes\tau\otimes Id_A"] \arrow[dd, bend right, "\mu_{A\otimes A}"'] \\ & A\otimes A\otimes A\otimes A \arrow[d,"\mu\otimes\mu"] \\ A \arrow[r,"\Delta_A"] & A\otimes A \end{tikzcd}$	$\begin{tikzcd} A\otimes A \arrow[r,"\mu"] \arrow[d,"\Delta_A\otimes\Delta_A"'] \arrow[dd, bend left, "\Delta_{A\otimes A}"] & A \arrow[dd,"\Delta_A"] \\ A\otimes A\otimes A\otimes A \arrow[d,"Id\otimes\tau\otimes Id"'] & \\ A\otimes A\otimes A\otimes A \arrow[r,"\mu_A\otimes\mu_A"] & A\otimes A \end{tikzcd}$
	$\begin{tikzcd} \mathbb{K} \arrow[r,"Id_{\mathbb{K}}"] \arrow[dd,"\eta_A"'] & \mathbb{K} \arrow[d,"\cong"',"\Delta_{\mathbb{K}}"] \arrow[dd, bend right, "\eta_{A\otimes A}"'] \\ & \mathbb{K}\otimes\mathbb{K} \arrow[d,"\eta_A\otimes\eta_A"] \\ A \arrow[r,"\Delta_A"] & A\otimes A \end{tikzcd}$	$\begin{tikzcd} \mathbb{K} \arrow[r,"\eta_A"] \arrow[d,"\Delta_{\mathbb{K}}"',"\cong"] & A \arrow[d,"\Delta_A"] \\ \mathbb{K}\otimes\mathbb{K} \arrow[r,"\eta_A\otimes\eta_A"] & A\otimes A \end{tikzcd}$
	$\begin{tikzcd} A\otimes A \arrow[r,"\varepsilon_A\otimes\varepsilon_A"] \arrow[d,"\mu_A"'] & \mathbb{K}\otimes\mathbb{K} \arrow[d,"\cong"',"\mu_{\mathbb{K}}"] \\ A \arrow[r,"\varepsilon"] & \mathbb{K} \end{tikzcd}$	$\begin{tikzcd} A\otimes A \arrow[r,"\mu_A"] \arrow[d,"\mu_A"'] \arrow[dd, bend left, "\varepsilon_A\otimes\varepsilon_A"] & A \arrow[dd,"\varepsilon_A"] \\ \mathbb{K}\otimes\mathbb{K} \arrow[d,"\mu_{\mathbb{K}}"',"\cong"] & \\ \mathbb{K} \arrow[r,"Id_{\mathbb{K}}"] & \mathbb{K} \end{tikzcd}$
	$\begin{tikzcd} \mathbb{K} \arrow[r,"Id_{\mathbb{K}}"] \arrow[d,"\eta_A"'] & \mathbb{K} \arrow[d,"\cong"',"\eta_{\mathbb{K}}"] \\ A \arrow[r,"\varepsilon_A"] & \mathbb{K} \end{tikzcd}$	$\begin{tikzcd} \mathbb{K} \arrow[r,"\eta_A"] \arrow[d,"\varepsilon_A"',"\cong"] & A \arrow[d,"\varepsilon_A"] \\ \mathbb{K} \arrow[r,"Id_{\mathbb{K}}"] & \mathbb{K} \end{tikzcd}$

HOPF ALGEBRA STRUCTURE

Definition A.13 (Hopf algebra)**.** *A Hopf algebra $\mathcal{H} \equiv (\mathcal{H}, \mu, \eta, \Delta, \varepsilon, S)$ is a $\mathbb{K}$-bialgebra $\mathcal{H}$ equipped with a linear map $S : \mathcal{H} \to \mathcal{H}$ called the* antipode *such that the following diagram commutes:*

$$\begin{tikzcd}
 & \mathcal{H}\otimes\mathcal{H} \arrow[rr, "S\otimes Id"] & & \mathcal{H}\otimes\mathcal{H} \arrow[dr, "\mu"] & \\
\mathcal{H} \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \mathbb{K} \arrow[rr, "\eta"] & & \mathcal{H} \\
 & \mathcal{H}\otimes\mathcal{H} \arrow[rr, "Id\otimes S"] & & \mathcal{H}\otimes\mathcal{H} \arrow[ur, "\mu"'] &
\end{tikzcd} \tag{282}$$

In other words, S is a left and right inverse of the linear map *Id* under the convolution product *$\star$:*

$$S \star Id = e = Id \star S\,. \tag{283}$$

Corollary A.2 (Any connected filtered bialgebra is a Hopf algebra (cf. [8], Ch. 4.3, Corr. 5))**.** *Let $\mathcal{A} \equiv (\mathcal{A}, \mu, \eta, \Delta, \varepsilon)$ be a connected filtered $\mathbb{K}$-bialgebra. Then it is also a Hopf algebra, $\mathcal{A} \equiv (\mathcal{A}, \mu, \eta, \Delta, \varepsilon, S)$, with antipode S given according to (281) (via the special case $\mathcal{A} = \mathcal{H}$ and $\varphi = Id$) by*

$$S(h) := Id^{\star\,-1}(h) = e(h) + \sum_{k=1}^{n}(e - Id)^{\star\,k}(h) \qquad \forall h \in \mathcal{H}^n \quad (n > 0)\,. \tag{284}$$

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HOPF ALGEBRA STRUCTURE

- Diagram algebra: ***filtration by number of connected components*** — for a basis diagram d(r,s,t), this would be r+s+t!
- The diagram algebra indeed fulfills all axioms of a Hopf algebra!

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- Diagram algebra: ***filtration by number of connected components*** — for a basis diagram d(r,s,t), this would be r+s+t!
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SUMMARY: THE FRAMEWORK SO FAR

- the rule algebra composition operations are defined via reduction of the rule diagram algebra composition operation

- ***result***: the rule algebras are ****-algebras !!!***

$$
\begin{tikzcd}
 & \mathcal{D} \arrow[loop above, "\alpha"] \arrow[dl, "\varphi"'] \arrow[dr, bend left, two heads, "\overline{\varphi}"] & & \text{Combinatorial Algebra} \\
\mathcal{U}(\mathcal{L}_{\mathcal{D}}) \arrow[rr, two heads, "\pi"] \arrow[ur, phantom, "\cong"] & & \mathcal{R} \arrow[ul, hook, bend left, "\psi"] & \text{Algebra} \\
 & \mathcal{L}_{\mathcal{D}} \arrow[ul, hook, "\iota"] \arrow[ur, hook', "\kappa"'] & & \text{Lie Algebra}
\end{tikzcd}
$$

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A HINT OF COMBINATORIAL PHYSICS

- ***coherent states*** (alternative basis for Fock space)
- ***formal power series***
- ***Blasiak et al.'s*** results via ***umbral calculus*** and ***Sheffer-type sequences***, leading e.g. to ***analytic solutions*** for e^{tH} for birth-death reactions

- Allan I Solomon et al. (2005): ***Combinatorial Physics, Normal Order and Model Feynman Graphs,*** *Symmetries in Science XI*. Springer, pp. 527–536
- Pawel Blasiak (2005): ***Combinatorics of boson normal ordering and some applications,*** arXiv preprint quant-ph/0507206
- Pawel Blasiak, Gerard HE Duchamp, et al. (2010): ***Combinatorial algebra for second-quantized Quantum Theory,*** Advances in Theoretical and Mathematical Physics 14.4, pp. 1209–1243

Nicolas Behr (2016): ***Combinatorial physics of graph rewriting,*** in preparation

CONCLUSION

- new concise framework, **first of its kind!**

- ***subsumes HW algebras as special case*** in a very precise manner

- natural framework of ***stochastic mechanics*** and ***combinatorial physics***

- ***bridgehead established*** between *computer science, combinatorics and theoretical physics*

OUTLOOK

- systematic search for ***analytically solvable GTSs***
    - applications of sophisticated GTSs applications such as ***kappa*** etc
- study of the intrinsic structure of ***network theory*** and ***game theory*** expressible in terms of the rule algebra

THANK YOU!

- Nicolas Behr, Vincent Danos, and Ilias Garnier (2016): ***Stochastic mechanics of graph rewriting,*** submitted to LICS 2016
- Nicolas Behr, Vincent Danos, Ilias Garnier, and Tobias Heindel (2016): ***The algebras of graph rewriting,*** in preparation
- Nicolas Behr (2016): ***Combinatorial physics of graph rewriting,*** in preparation