Compositionality of Rewriting Rules with Conditions

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We extend the notion of compositional associative rewriting as recently studied in the rule algebra framework literature to the setting of rewriting rules with conditions. Our methodology is category-theoretical in nature, where the definition of rule composition operations encodes the non-deterministic sequential concurrent application of rules in Double-Pushout (DPO) and Sesqui-Pushout (SqPO) rewriting with application conditions based upon $\mathcal{M}$-adhesive categories. We uncover an intricate interplay between the category-theoretical concepts of conditions on rules and morphisms, the compositionality and compatibility of certain shift and transport constructions for conditions, and thirdly the property of associativity of the composition of rules.

NEW DOUBLE-CATEGORICAL FRAMEWORK !

Fundamentals of Compositional Rewriting Theory★

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CASE STUDY: REWRITING OF SIMPLE GRAPHS

V1:

$$\underline{\mathsf{SGraph}} := \underline{\mathsf{Set}} /\!/ \Delta := \begin{cases} (\underline{\mathsf{SGraph}})_0 := \left\{ E \overset{u}{\hookrightarrow} V \times V \in \mathrm{mono}(\underline{\mathsf{Set}}) \right\} \\ (\underline{\mathsf{SGraph}})_1 := \left\{ u \xrightarrow{(\varepsilon,\nu)} u' \;\middle|\; E \xrightarrow{\varepsilon} E',\ V \xrightarrow{\nu} V',\ (\Delta\nu) \circ u = u' \circ \varepsilon \right\} \end{cases}$$

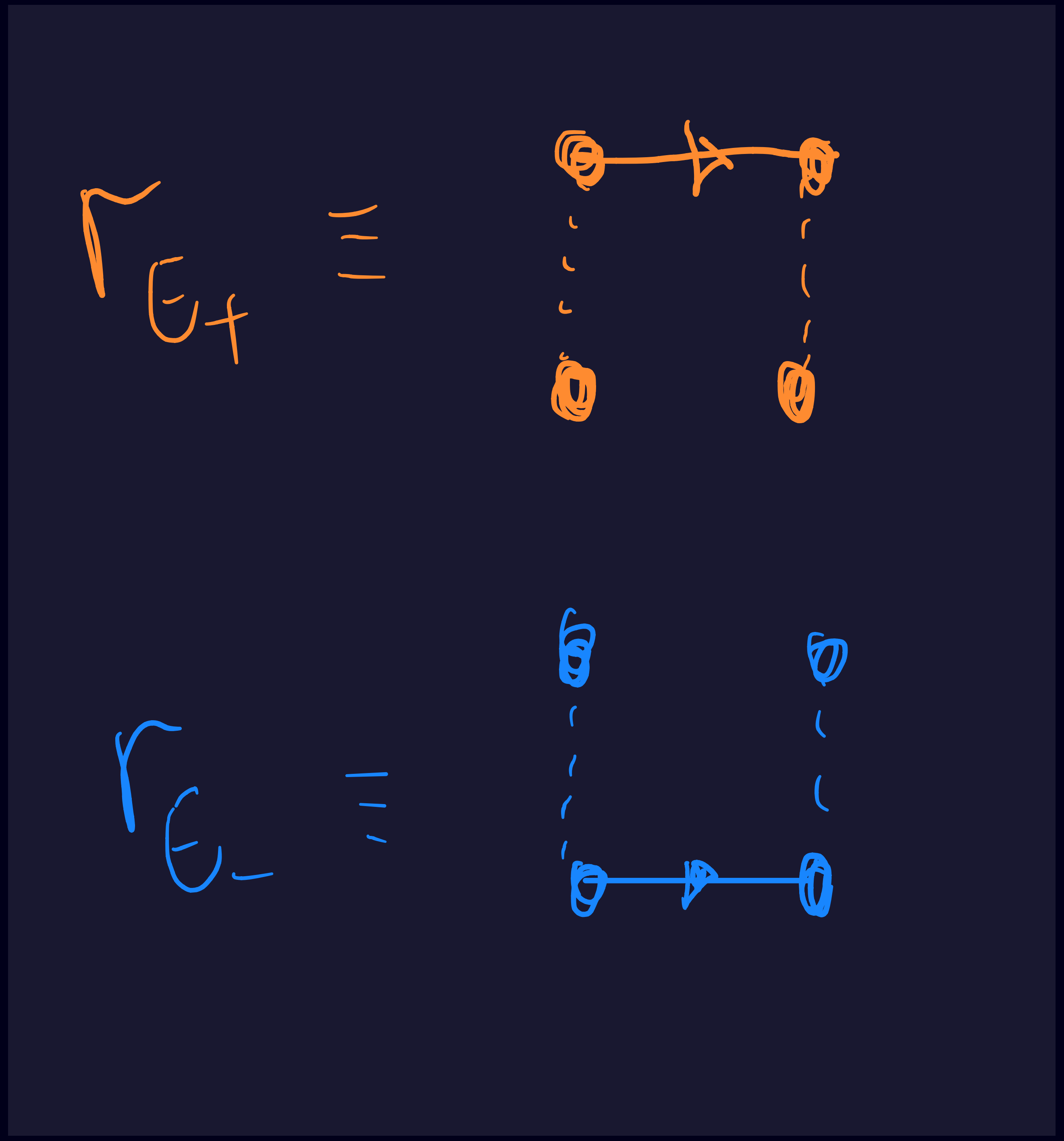
$$\begin{tikzcd} E \arrow[r, hook, "u"] \arrow[d, "\varepsilon"'] & V \times V \arrow[d, "\Delta\nu = \nu\times\nu"] & V \arrow[l, "\Delta"'] \arrow[d, "\nu"] \\ E' \arrow[r, hook, "u'"] & V' \times V' & V' \arrow[l, "\Delta"] \end{tikzcd}$$

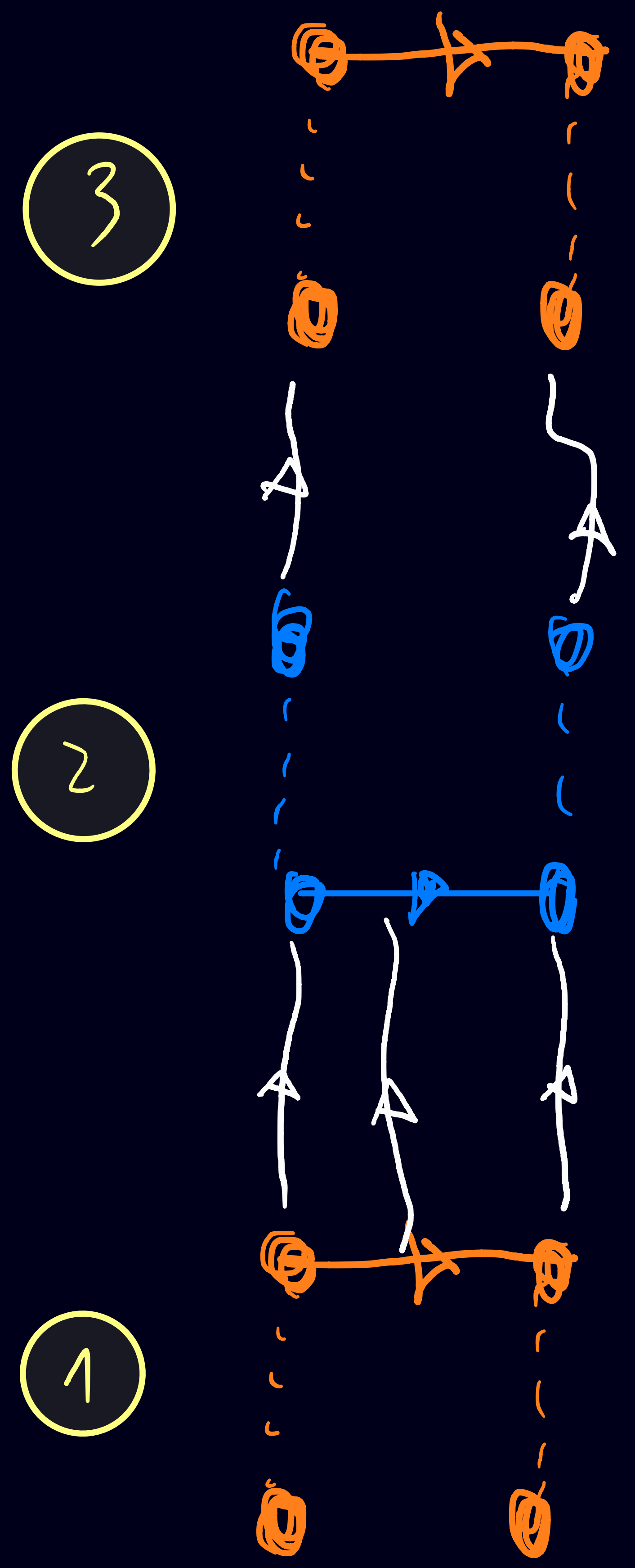
$\underline{V2}$: via RESTRICTION of multi-graphs:

$$\underline{\mathrm{sGraph}} := \underline{\mathrm{Graph}}\Big|_{\mathcal{C}_{sG}} := \begin{cases} (\underline{\mathrm{sGraph}})_0 := \{ E \xrightarrow{u} V \times V \mid u \vDash \mathcal{C}_{sG} \} \\ (\underline{\mathrm{sGraph}})_1 := (\underline{\mathrm{Graph}})_1 \Big|_{(\underline{\mathrm{sGraph}})_0} \end{cases}$$

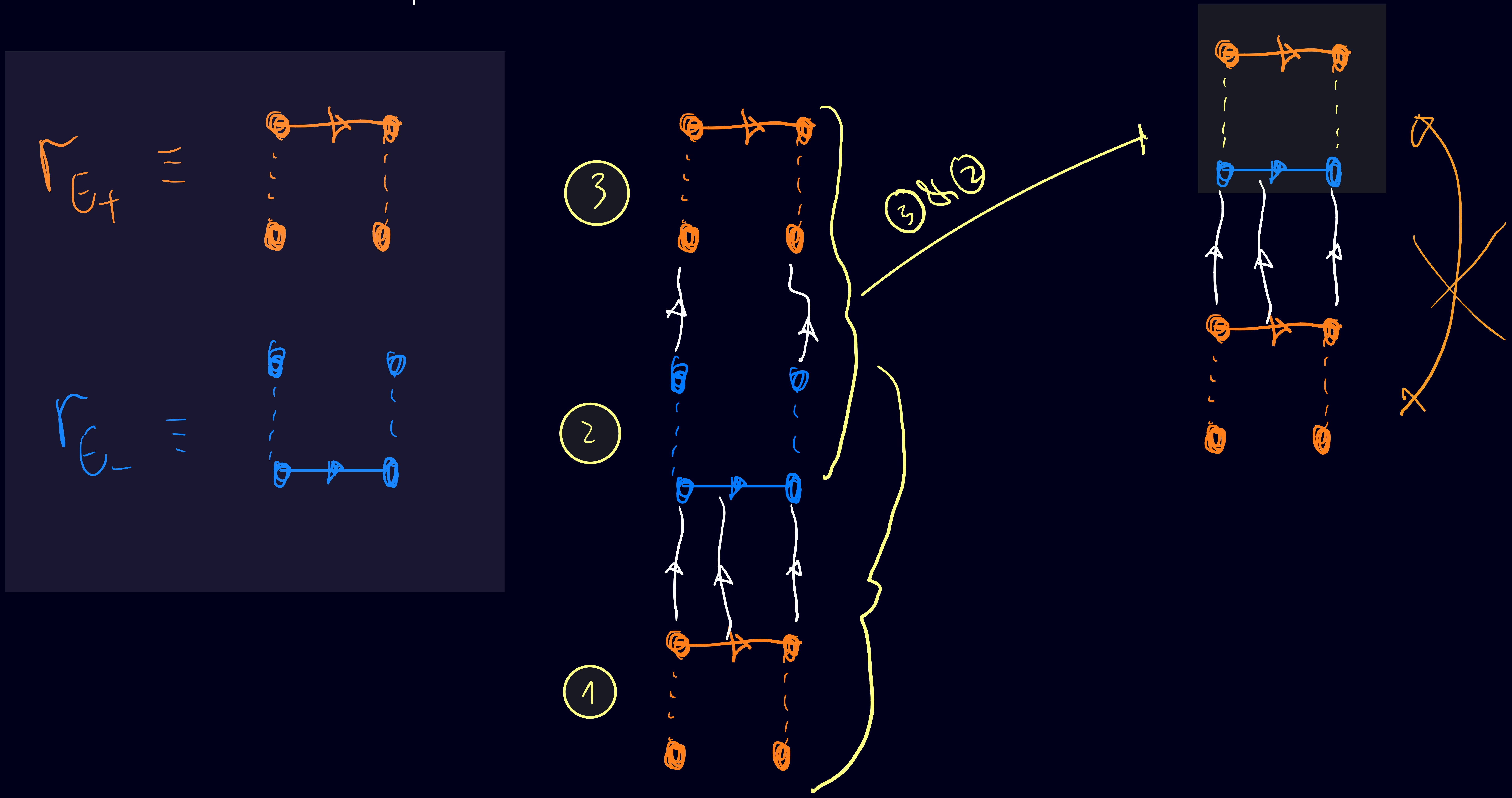
$$\mathcal{C}_{sG} := \neg\exists\left(\underset{v_1}{\bullet} \;\; \underset{v_2}{\bullet} \hookrightarrow \underset{v_1}{\bullet} \rightrightarrows \underset{v_2}{\bullet} \right)$$

↳ Consider operations of CREATION and DELETION of edges:

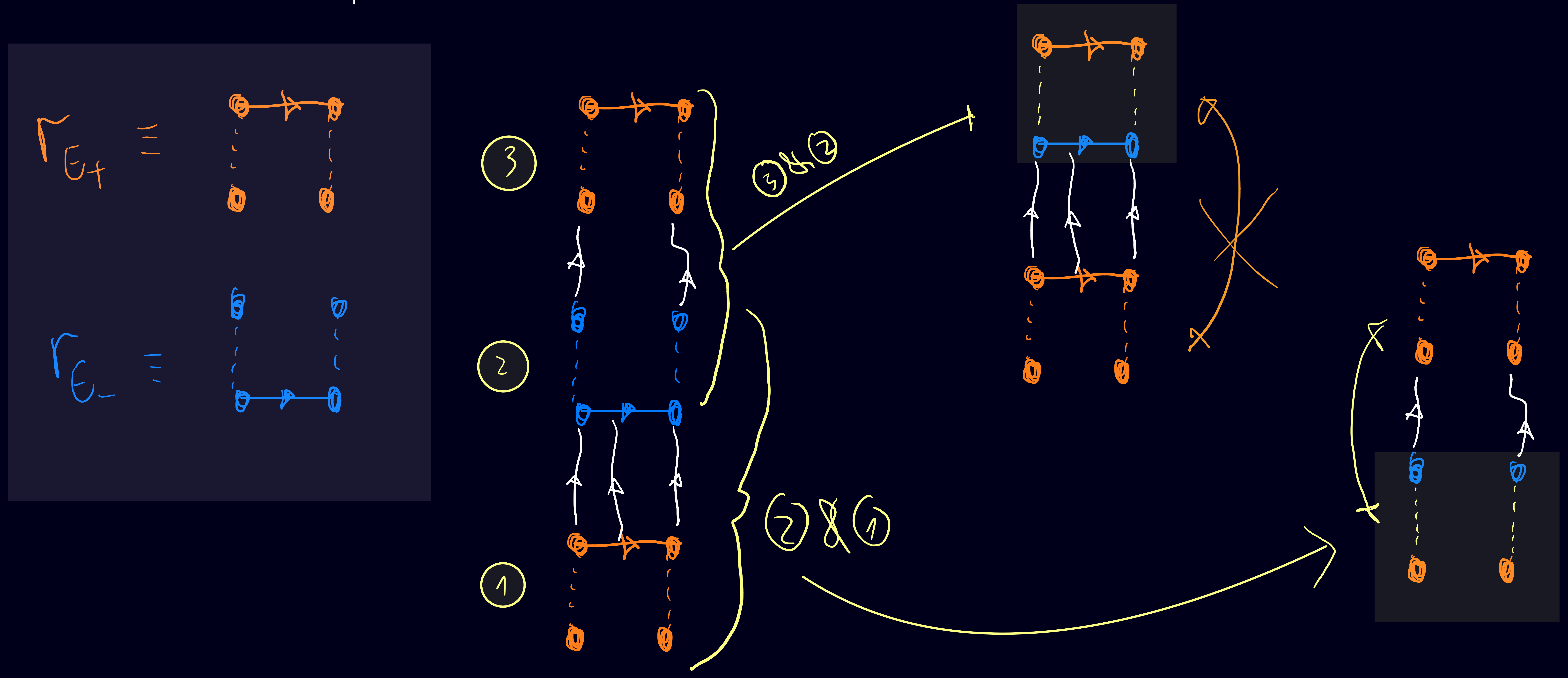




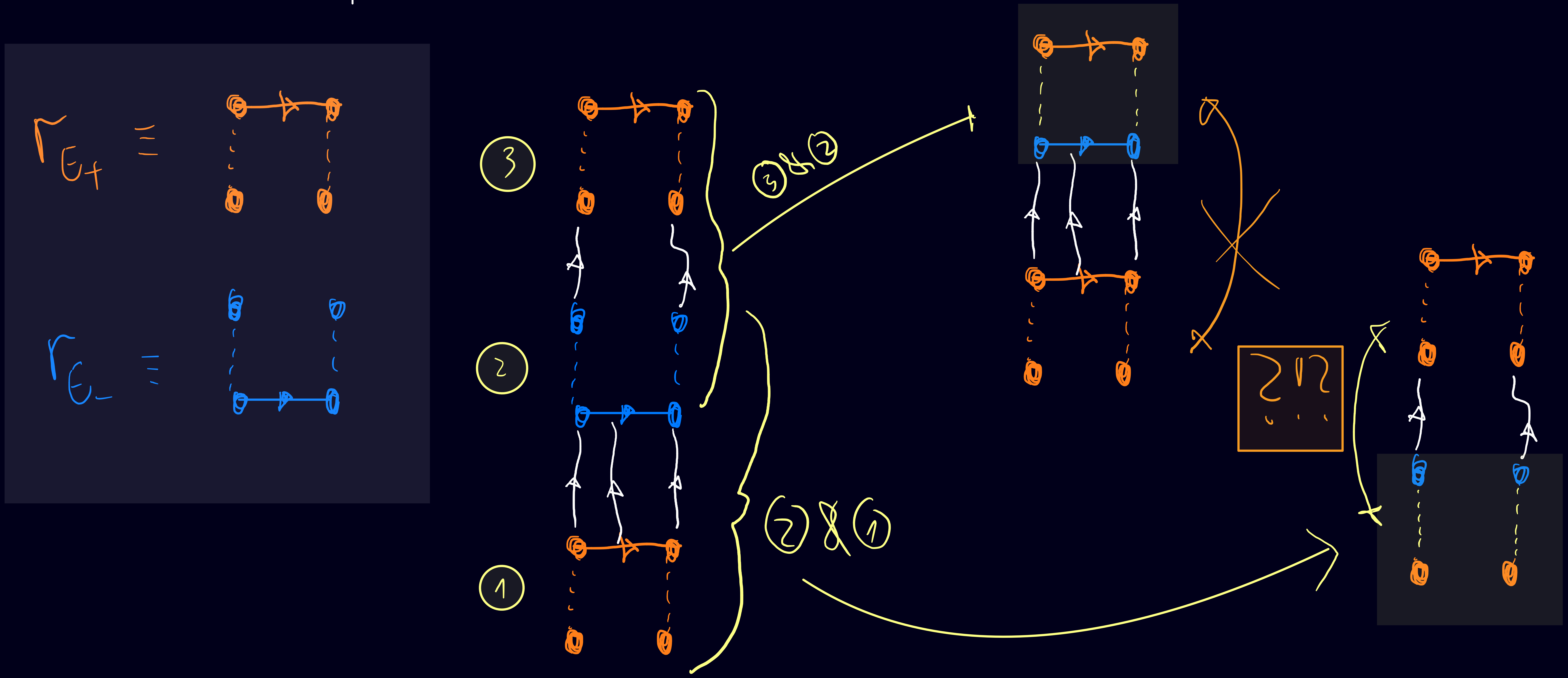
↳ Consider operations of CREATION and DELETION of edges:



↳ Consider operations of CREATION and DELETION of edges:



↳ Consider operations of CREATION and DELETION of edges:



⇒ SURPRISING ISSUE with $\underline{\mathrm{SGraph}} := \underline{\mathrm{Set}} // \Delta$:

DPO-/SqPO-rewriting in $\underline{\mathrm{SGraph}}$ for MONIC matches

IS NOT COMPOSITIONAL !?!

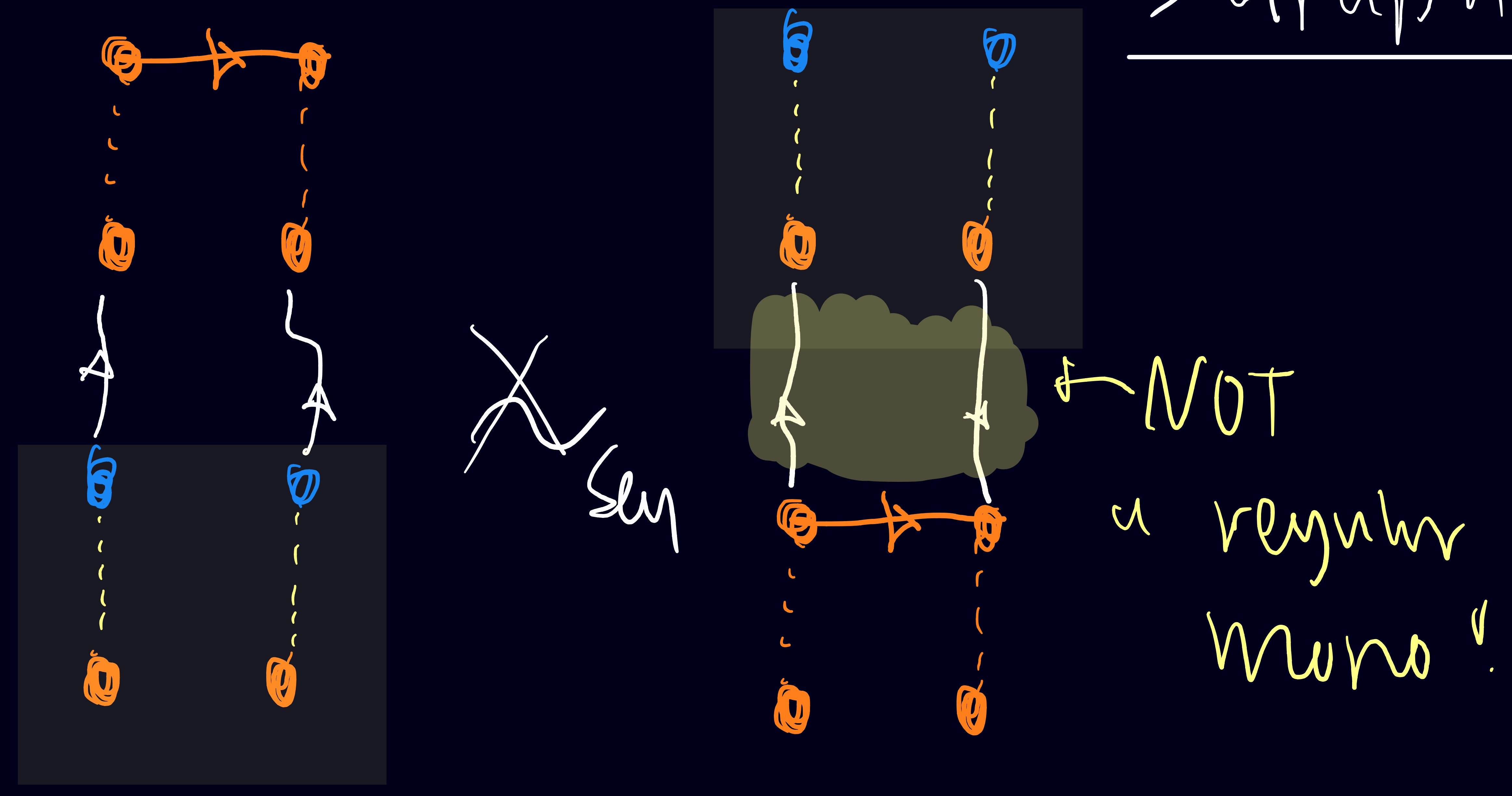
BUT: —"— for REGULAR MONIC matches

edge-reflecting

IS compositional!

?!?!?!

NOTE: the diagram below right does NOT exist in $\underline{\mathbf{SGraph}}\big|_{rm\text{-}matches}$ NOR in $\underline{\mathbf{sGraph}}$ (for rules w/ constraint-preserving conditions)

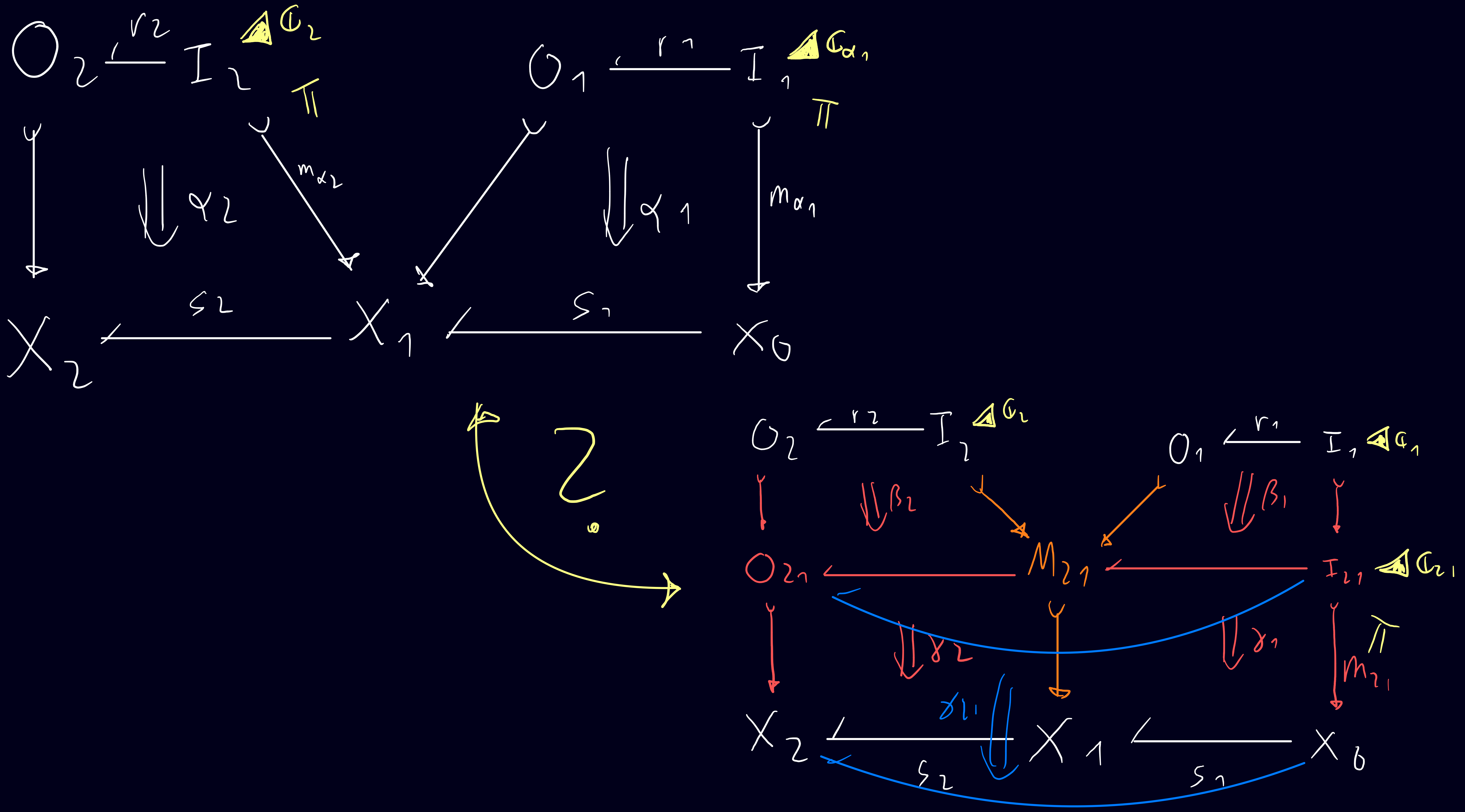


$$
\begin{tikzcd}
O \arrow[d, tail, "n_\alpha"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, tail, "m_\alpha"] \\
r_\alpha(X) & X \arrow[l, "r_\alpha"]
\end{tikzcd}
\quad := \quad
\begin{tikzcd}
O \arrow[d, tail, "n_\alpha"'] \arrow[dr, phantom, "PO"] & U \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, tail, "u_\alpha"] \arrow[dr, phantom, "(*)"] & I \arrow[d, tail, "m_\alpha"] \\
r_\alpha(X) & U_\alpha \arrow[l] \arrow[r] & X
\end{tikzcd}
$$

DPO: $(*) = PO$

SqPO: $(*) = FPC$

$$
\begin{tikzcd}
O \arrow[d, tail, "n_\alpha"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, tail, "m_\alpha"] & \blacktriangleleft \mathbb{C}_I \quad \pi \\
r_\alpha(X) & X \arrow[l, "r_\alpha"]
\end{tikzcd}
$$


O2 r2 I2 O1 r1 I1
mα2 α2 α1 mα1
X2 s2 X1 s1 X0
?
O2 r2 I2 O1 r1 I1
β2 β1
O21 M21 I21
γ2 γ1 m21
γ21
X2 s2 X1 s1 X0

PLAN

I. CALCULUS OF CONDITIONS

II. COMPOSITIONALITY

III. REFINEMENTS

PRELIMINARIES

STABLE SYSTEM OF MONICS $\mathcal{M} \subseteq mono(\mathbf{C})$ — class satisfying

(i) $\forall\ g \circ f \in mor(\mathbf{C})$: $f \in \mathcal{M} \wedge g \in \mathcal{M} \Rightarrow g \circ f \in \mathcal{M}$

(ii) $\forall\ g \circ f \in mor(\mathbf{C})$: $g \in \mathcal{M} \wedge g \circ f \in \mathcal{M} \Rightarrow f \in \mathcal{M}$

(iii) $\mathbf{C}$ HAS PULLBACKS ALONG $\mathcal{M}$-morphisms

(iv) $\mathcal{M}$ is STABLE UNDER PULLBACKS

NOTATION:

$\rightarrowtail$

PRELIMINARIES

$\mathcal{M}$ — STABLE SYSTEM OF MONICS in $\mathbf{C}$

$\varnothing \in obj(\mathbf{C})$ is a STRICT $\mathcal{M}$-INITIAL OBJECT if

(i) $\forall\ X \in obj(\mathbf{C}):\ \exists!\ \varnothing \xrightarrow{i_X} X \in \mathcal{M}$

(ii) $\forall\ X \xrightarrow{f} \varnothing \in mor(\mathbf{C}):\ f \in iso(\mathbf{C})$

CALCULUS OF CONDITIONS À LA HABEL & PENNEMANN

$\mathcal{M}$ – stable system of monics in some category $\mathcal{C}$

(NESTED) CONDITIONS $\mathbb{C}_X$ (for $X \in obj(\mathcal{C})$) are recursively defined as follows:

(i) $\mathbb{C}_X = true_X$ is a condition

(ii) $\mathbb{C}_X = \exists(X \xrightarrow{m} Y, \mathbb{C}_Y)$ is a condition

(iii) $\mathbb{C}_X$ is a condition $\Rightarrow \neg\mathbb{C}_X$ is a condition

(iv) $\mathbb{C}_X^{(1)}, \mathbb{C}_X^{(2)}$ are conditions $\Rightarrow \mathbb{C}_X^{(1)} \wedge \mathbb{C}_X^{(2)}$ is a condition

Calculus of conditions à la Habel & Pennemann

Satisfaction of conditions: $\forall\, X \xrightarrow{h} Z \in \mathcal{M}$

(i) $h \models \mathsf{true}_X$ (Notation: $\mathsf{true} \equiv \mathsf{true}_X$)

(ii) $h \models \exists(X \xrightarrow{f} Y, \mathsf{c}_Y) \;:\Leftrightarrow\; \exists\, Y \xrightarrow{g} Z : h = g \circ f \wedge g \models \mathsf{c}_Y$

(iii) $h \models \neg \mathsf{c}_X \Leftrightarrow h \not\models \mathsf{c}_X$

(iv) $h \models \mathsf{c}_X^{(1)} \wedge \mathsf{c}_X^{(2)}$

$\Leftrightarrow h \models \mathsf{c}_X^{(1)} \wedge h \models \mathsf{c}_X^{(2)}$

$$\begin{tikzcd}
\mathsf{c}_X \arrow[r, phantom, "\triangleright"] & X \arrow[r, "f"] \arrow[dr, "h"'] & Y \arrow[d, dotted, "g"] & \mathsf{c}_Y \arrow[l, phantom, "\triangleleft"] \\
 & & Z &
\end{tikzcd}$$

Calculus of Conditions à la Habel & Pennemann

$\emptyset$ — strict $\mathcal{M}$-initial object

Constraints on objects := conditions on $\emptyset$

$$X \vDash c_\emptyset \;:=\; (\emptyset \xrightarrow{i_X} X) \vDash c_\emptyset$$

Examples:

- $\neg\exists(\emptyset \rightarrowtail \bullet\;\bullet,\ \exists(\bullet\;\bullet \rightarrowtail \bullet \rightleftarrows \bullet,\ \mathsf{true}))$
- $\forall(\emptyset \rightarrowtail \circlearrowleft\!\bullet,\ \exists(\circlearrowleft\!\bullet \rightarrowtail \circlearrowleft\!\bullet \rightarrow \bullet,\ \mathsf{true}))$

$$= \neg\exists(\emptyset \rightarrowtail \circlearrowleft\!\bullet,\ \neg\exists(\circlearrowleft\!\bullet \rightarrowtail \circlearrowleft\!\bullet \rightarrow \bullet,\ \mathsf{true}))$$

CALCULUS OF CONDITIONS À LA HABEL & PENNEMANN

EQUIVALENCE OF CONDITIONS:

$$\mathbb{C}_X^{(1)} \equiv \mathbb{C}_X^{(2)} :\Leftrightarrow \forall\, X \xrightarrow{f} Y : f \vDash \mathbb{C}_X^{(1)} \Leftrightarrow f \vDash \mathbb{C}_X^{(2)}$$

NOTE: König et al. also define

$$\mathbb{C}_X^{(1)} \vDash \mathbb{C}_X^{(2)} :\Leftrightarrow \forall\, X \xrightarrow{f} Y : f \vDash \mathbb{C}_X^{(1)} \Rightarrow f \vDash \mathbb{C}_X^{(2)}$$

↪ Category $\mathrm{cond}(\mathbf{C}, \mathcal{M})$ OF CONDITIONS...

SHIFT CONSTRUCTION

$$\begin{tikzcd}
\mathfrak{C}_X \triangleright X \arrow[r, tail, "f"] \arrow[dr, tail, "h"'] & Y \triangleleft \mathrm{Shift}(f, \mathfrak{C}_X) \arrow[d, dotted, hook, "g"] \\
& Z
\end{tikzcd}$$

$\forall f, g, h \in \mathcal{M}: \; h = g \circ f: \quad h \vDash \mathfrak{C}_X \Leftrightarrow g \vDash \mathrm{Shift}(f, \mathfrak{C}_X)$

THEOREM: If $\mathcal{C}$ is an $\mathcal{M}$-adhesive category, Shift exists.

CONSTRUCTION:

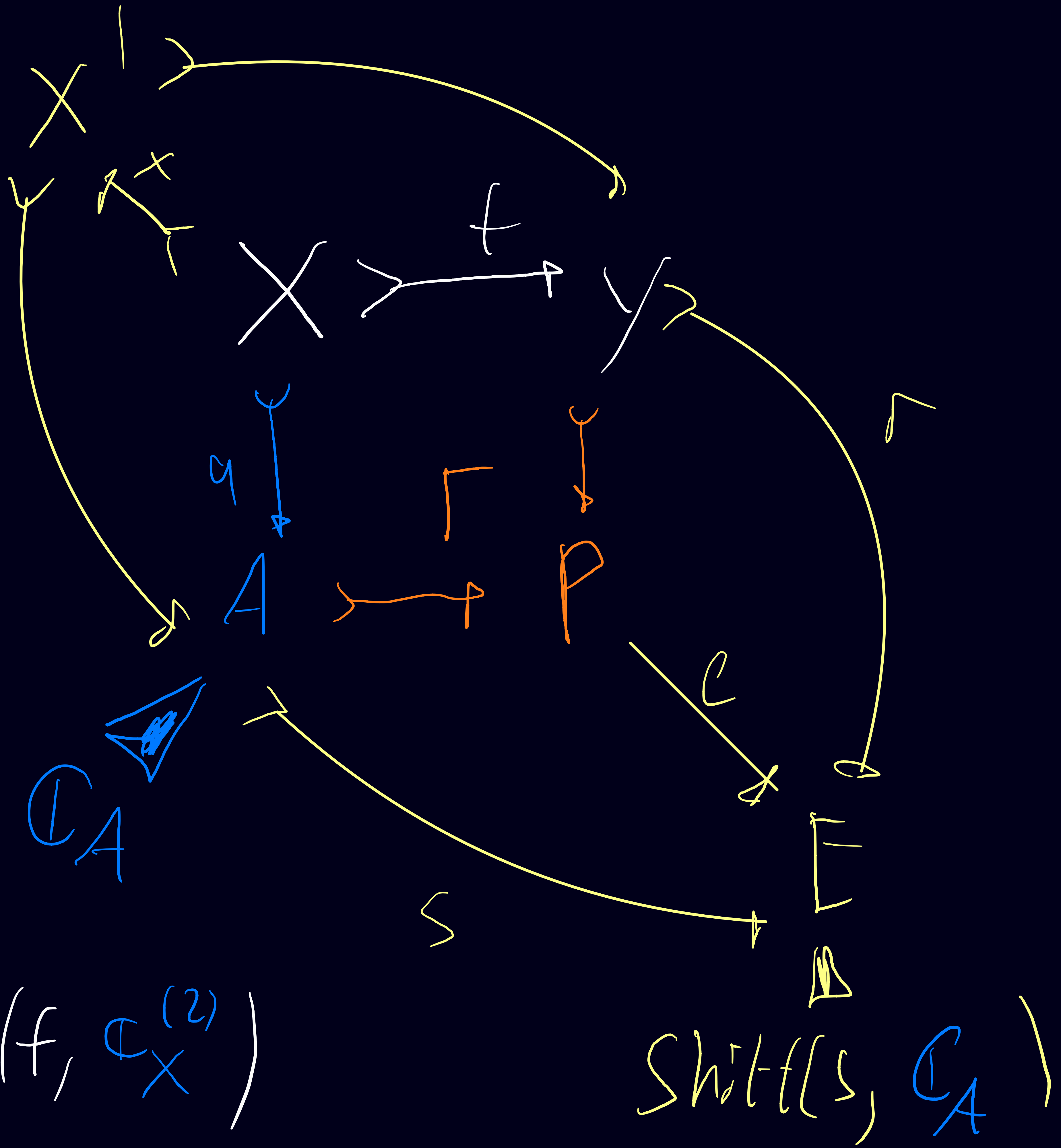
(i) $\mathrm{Shift}(X \xrightarrow{t} Y, \mathsf{true}_X) := \mathsf{true}_Y$

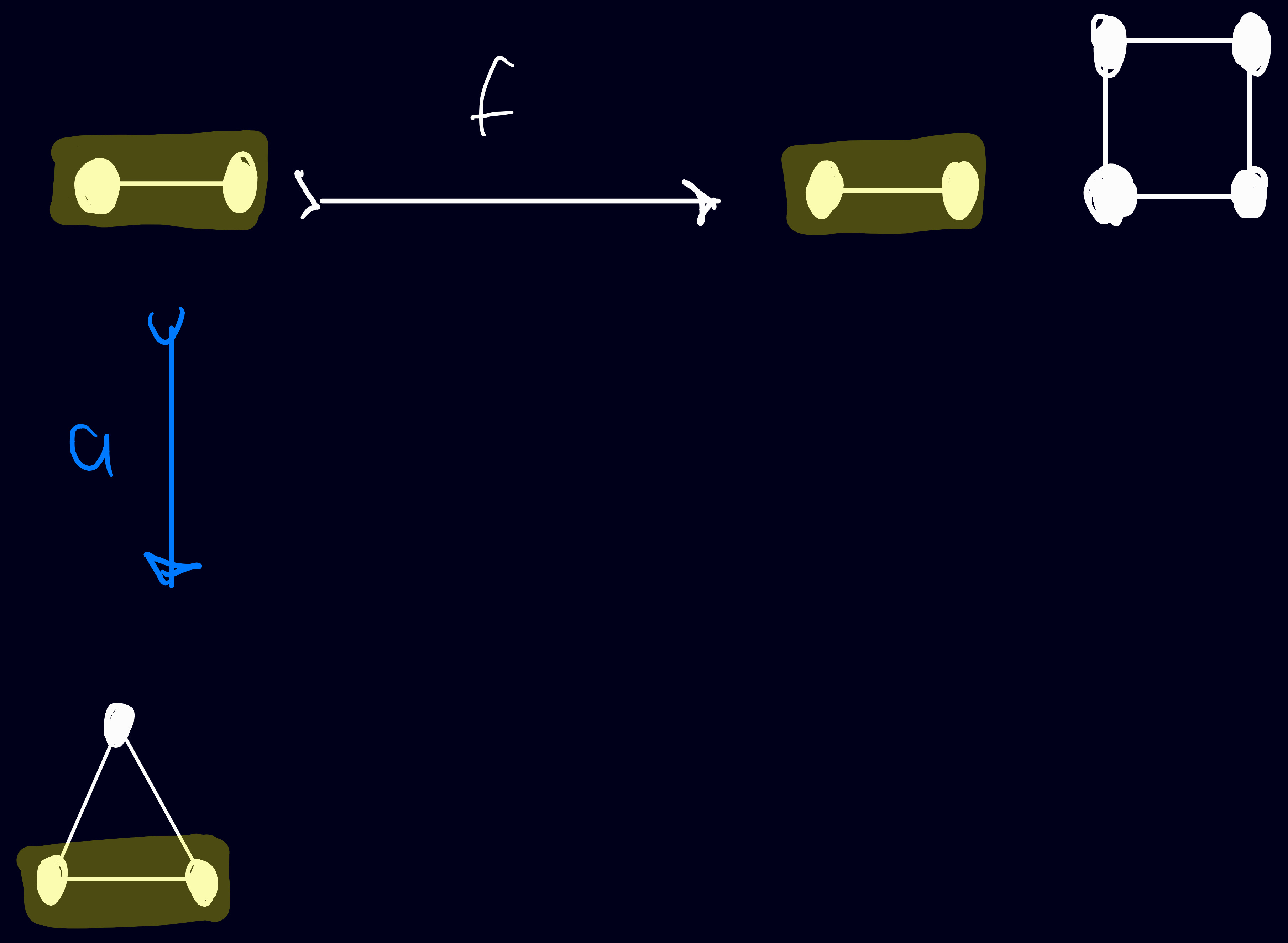
(ii) $\mathrm{Shift}(X \xrightarrow{t} Y, \exists(X \xrightarrow{a} A, \mathfrak{c}_A))$

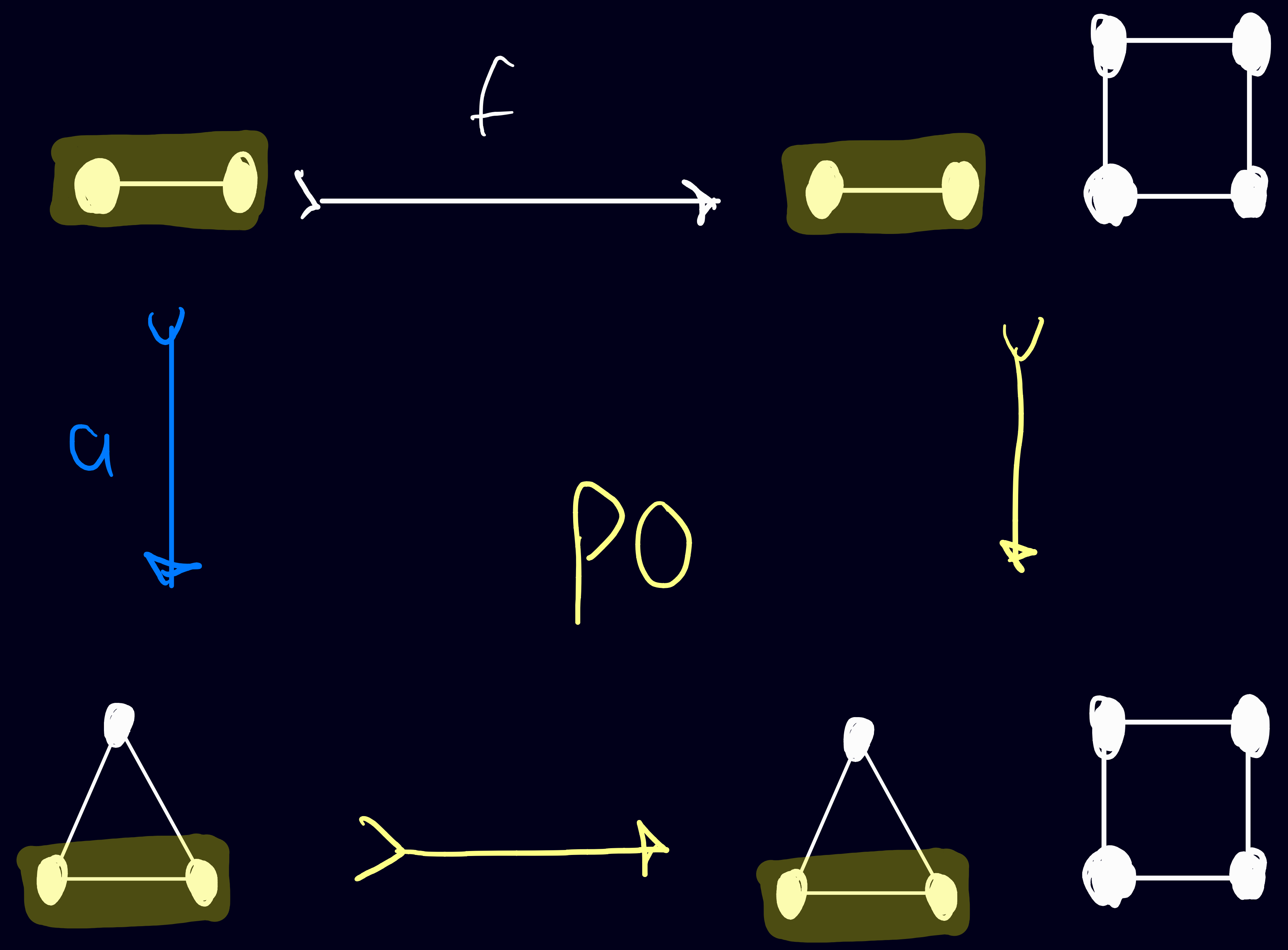
$$:= \bigvee_{(r,s)} \exists(r, \mathrm{Shift}(s, \mathfrak{c}_A))$$

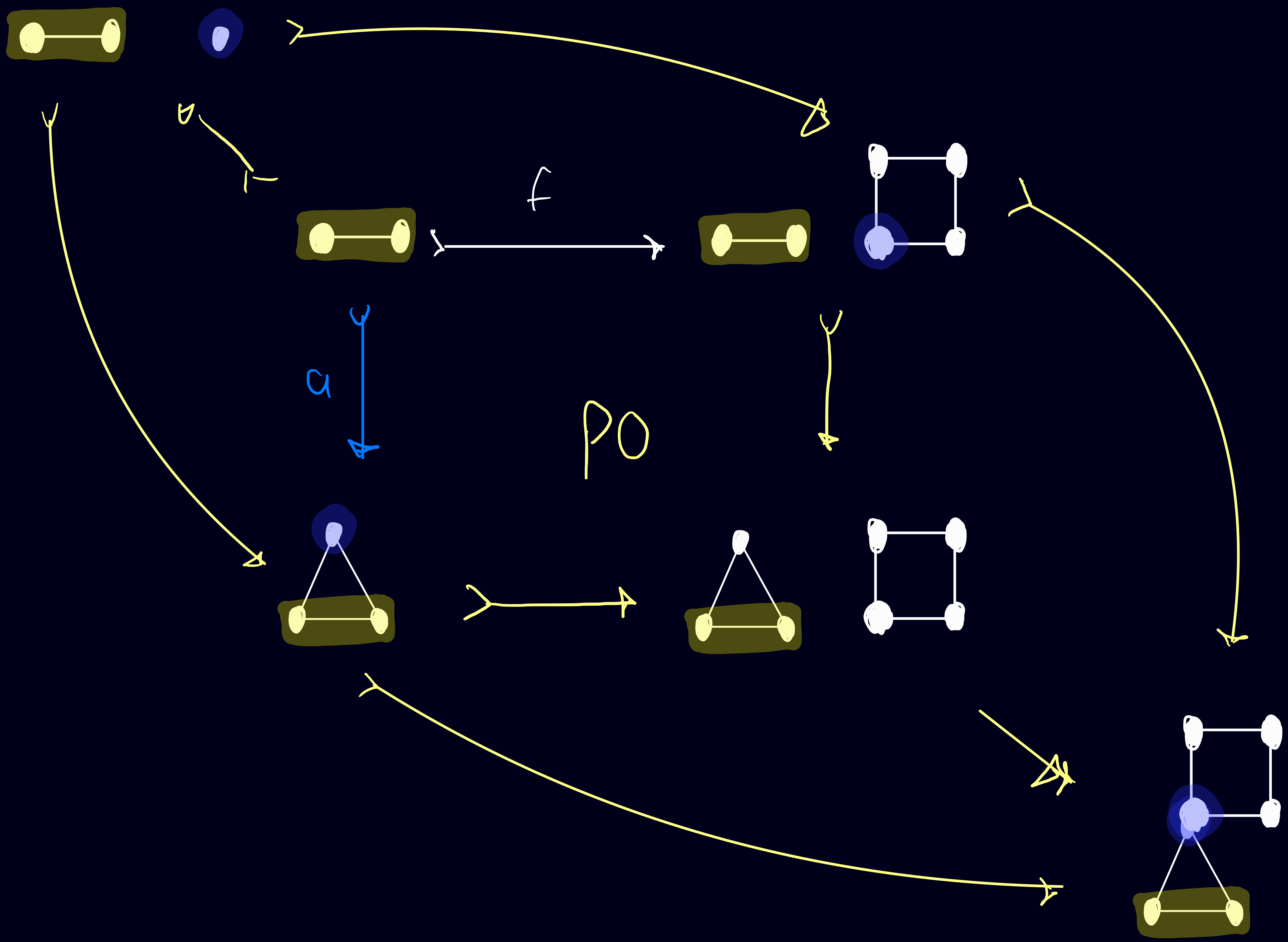
(iii) $\mathrm{Shift}(t, \neg \mathfrak{c}_X) := \neg\, \mathrm{Shift}(t, \mathfrak{c}_X)$

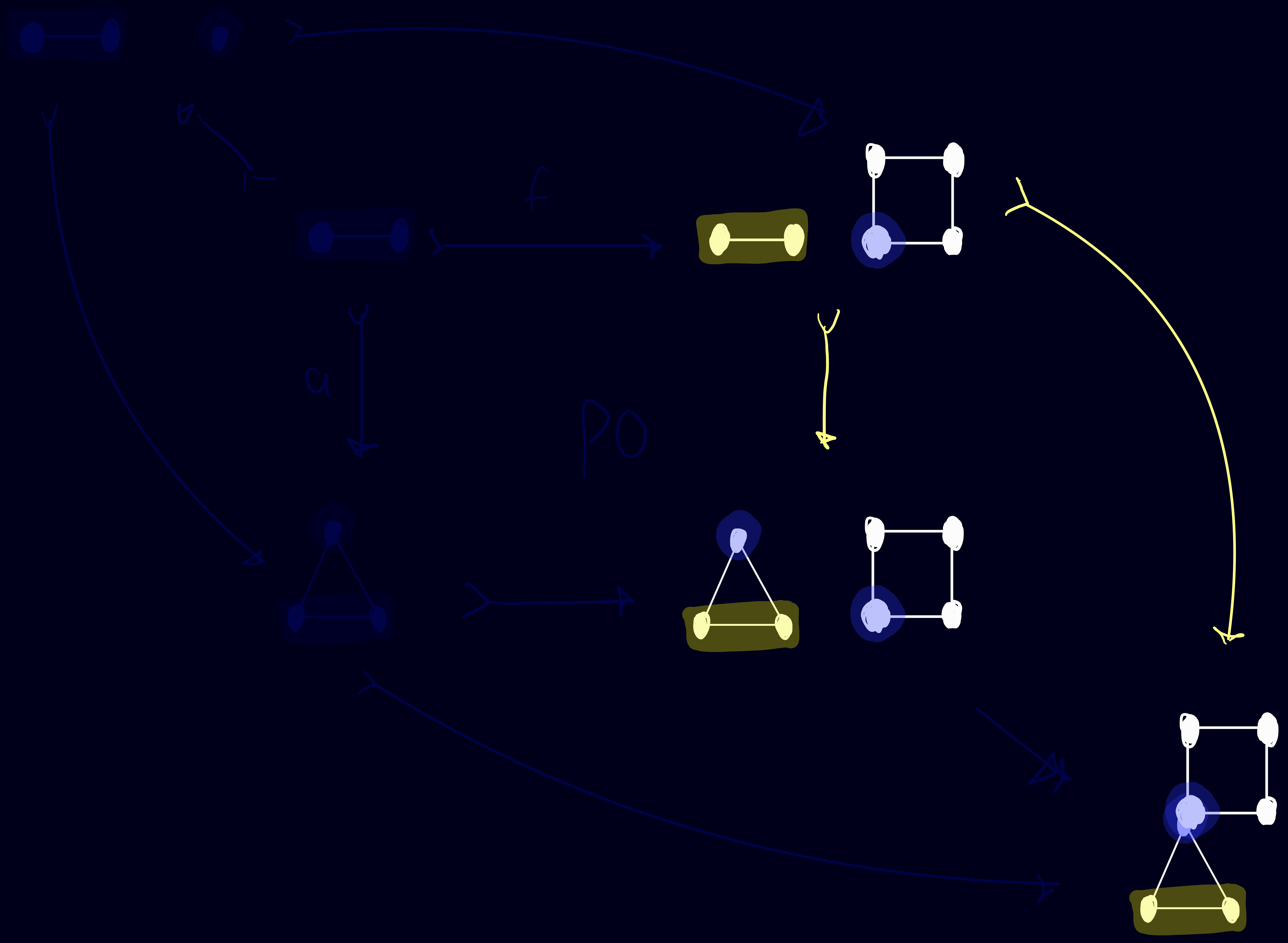
(iv) $\mathrm{Shift}(t, \mathfrak{c}_X^{(1)} \wedge \mathfrak{c}_X^{(2)}) := \mathrm{Shift}(t, \mathfrak{c}_X^{(1)}) \wedge \mathrm{Shift}(t, \mathfrak{c}_X^{(2)})$




f
a


f
a
PO


f
a
PO


f
a
PO

Properties of the Shift construction

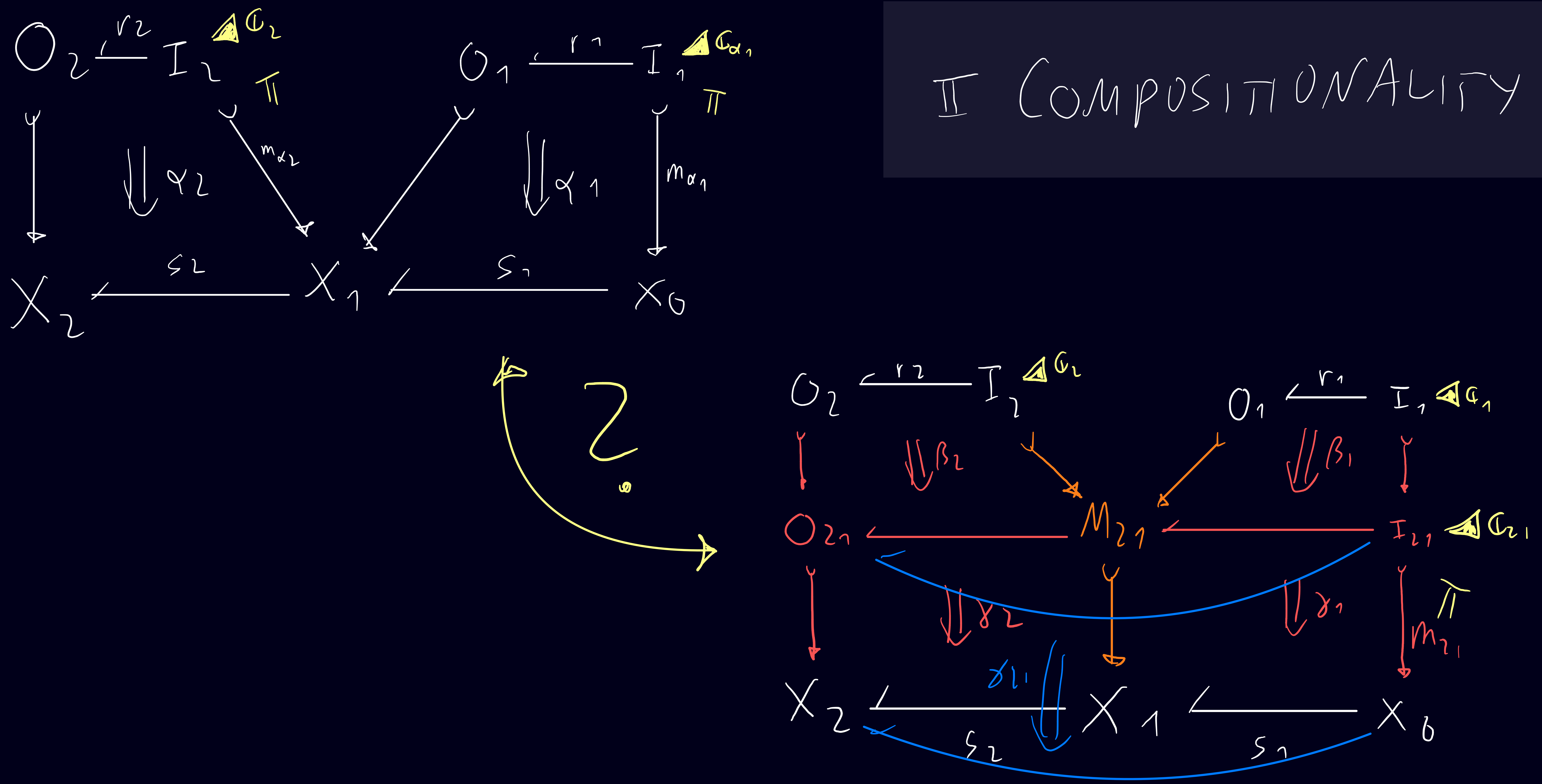
(I) $\forall\; A \xrightarrow[\cong]{\alpha} A' \in \mathrm{iso}(\mathbf{C}),\; \mathfrak{c}_A \in \mathrm{cond}(\mathbf{C}, \mathcal{M}):\; \mathrm{Shift}(\alpha, \mathfrak{c}_A) \equiv \mathfrak{c}_A$

(II) $\forall\; A \overset{f}{\rightarrowtail} B,\; B \overset{g}{\rightarrowtail} C \in \mathcal{M},\; \mathfrak{c}_A \in \mathrm{cond}(\mathbf{C}, \mathcal{M}):\; \mathrm{Shift}(g, \mathrm{Shift}(f, \mathfrak{c}_A)) \equiv \mathrm{Shift}(g \circ f, \mathfrak{c}_A)$

$$\begin{tikzcd}
 & & \mathrm{Shift}(f, \mathfrak{c}_A) \arrow[d, phantom, "\triangledown"] & & \mathrm{Shift}(g, \mathrm{Shift}(f, \mathfrak{c}_A)) \arrow[d, phantom, "\cong"] \\
\mathfrak{c}_A & A \arrow[l, phantom, "\triangleleft"] \arrow[r, tail, "f"] \arrow[dr, "h"'] & B \arrow[r, tail, "g"] \arrow[d, dashed, "p"'] & C \arrow[dl, dashed, "q"] \arrow[ur, phantom, "\triangleleft"] \arrow[r, phantom, "\triangleleft"] & \mathrm{Shift}(g \circ f, \mathfrak{c}_A) \\
 & & X & &
\end{tikzcd}$$

$h = q \circ g \circ f$
$= p \circ f$

II COMPOSITIONALITY



RECAP

- PULLBACKS:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\;:\Longleftrightarrow\; \forall\;
\begin{tikzcd}
X \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& A \arrow[r] \arrow[d] & B \arrow[d] \\
& C \arrow[r] & D
\end{tikzcd}
\;:\;
\begin{tikzcd}
X \arrow[drr, bend left] \arrow[ddr, bend right] \arrow[dr, dashed, "\exists!"] & & \\
& A \arrow[r] \arrow[d] & B \arrow[d] \\
& C \arrow[r] & D
\end{tikzcd}
$$

- PUSHOUTS:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\;:\Longleftrightarrow\; \forall\;
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D & \\
& & Y
\end{tikzcd}
\;:\;
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!"] & \\
& & Y
\end{tikzcd}
$$

- FINAL PULLBACK COMPLEMENT (FPC):

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{FPC}"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\quad :\Longleftrightarrow \quad \forall
\begin{tikzcd}
P \arrow[dr] \arrow[drr, bend left] \arrow[ddd] & & \\
& A \arrow[r] \arrow[d] & B \arrow[d] \\
& C \arrow[r] & D \\
Q \arrow[urr, bend right] & &
\end{tikzcd}
\; : \;
\begin{tikzcd}
P \arrow[dr] \arrow[drr, bend left] \arrow[ddd] & & \\
& A \arrow[r] \arrow[d] & B \arrow[d] \\
& C \arrow[r] & D \\
Q \arrow[urr, bend right] \arrow[ur, dashed, "\exists!"] & &
\end{tikzcd}
$$

s.th.
$$
\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & B \arrow[d] \\
Q \arrow[r] & D
\end{tikzcd}
$$

s.th.
$$
\begin{tikzcd}
P \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{PB}"] & A \arrow[d] \\
Q \arrow[r, dashed] & C
\end{tikzcd}
$$

Double-Pushout (DPO) and Sesqui-Pushout (SqPO) Semantics ("linear" version)

$\mathcal{M}$ — stable system of monics

$\mathcal{C}$ — $\mathcal{M}$-adhesive category

$$
\begin{tikzcd}
O \arrow[d] & I \arrow[l, "r"'] \arrow[d] \arrow[dl, phantom, "\Downarrow\alpha"] & \triangleleft\, \mathcal{C}_I \\
Y & X \arrow[l, "r_\alpha"]
\end{tikzcd}
\quad ::= \quad
\begin{tikzcd}
O \arrow[d, "n_\alpha"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "(*)"] & I \arrow[d, "m_\alpha"] & \triangleleft\, \mathcal{C}_I \\
Y & \bar{X}_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X
\end{tikzcd}
$$

DPO: $(*) = PO$

SqPO: $(*) = FPC$

$\mathcal{M}$-MULTISUMS

$$\forall\, A, B \in obj(\mathbf{C}) : \exists\, \Sigma^{\vee}_{(A,B)} = \left\{ \begin{tikzcd} A \arrow[dr, tail, "\alpha_j"'] & & B \arrow[dl, tail, "\beta_j"] \\ & M_j & \end{tikzcd} \right\}_{j \in J_{A,B}} :$$

$$\begin{tikzcd} A \arrow[dr, tail, bend right, "a"'] & & B \arrow[dl, tail, bend left, "b"] \\ & Z & \end{tikzcd} : \exists!\, j \in J_{A,B} : \begin{tikzcd} A \arrow[r, tail, "\alpha_j"] \arrow[dr, tail, bend right, "a"'] & M_j \arrow[d, dotted, "\exists: m"] & B \arrow[l, tail, "\beta_j"'] \arrow[dl, tail, bend left, "b"] \\ & Z & \end{tikzcd}$$

$\mathcal{M}$-MULTISUMS

CONSTRUCTION:

$$\forall\, A, B \in \mathrm{obj}(\mathcal{C}):\quad \Sigma^{\gamma}_{A,B} := \left\{ \begin{tikzcd} A \arrow[dr, tail, "\alpha_j"'] & & B \arrow[dl, tail, "\beta_j"] \\ & M_j & \end{tikzcd} \;\middle|\; \exists\; \begin{tikzcd} & X_j \arrow[dl, tail, "\gamma_j"'] \arrow[dr, tail, "\delta_j"] & \\ A \arrow[dr, tail, "\alpha_j"'] & \text{PO} & B \arrow[dl, tail, "\beta_j"] \\ & M_j & \end{tikzcd} \right\}$$

Pick one cospan per iso. class

Fibrational Structures

Theorem: Let $\mathcal{C}$ be an $\mathcal{M}$-adhesive category.

$$
\forall \begin{tikzcd} O & I \arrow[l, "r"'] \\ Y \arrow[from=1-1, tail, "n"'] & \end{tikzcd}
: \exists\, \mathsf{T}(r,n) = \left\{ \begin{tikzcd} O \arrow[d, tail, "n"'] \arrow[dr, phantom, "\Downarrow \alpha_j"] & I \arrow[l, "r"'] \arrow[d, "n^*_j"] \\ Y & X_j \arrow[l] \end{tikzcd} \right\}_{j \in \mathcal{J}_{r,n}}
: \forall \begin{tikzcd} O \arrow[d, tail, "n"'] \arrow[dd, bend right] \arrow[ddr, phantom, "\Downarrow \gamma"] & I \arrow[l, "r"'] \arrow[dd] \\ Y \arrow[d] & \\ Z & X' \arrow[l] \end{tikzcd}
: \exists!\, j \in \mathcal{J}_{r,n} :
\begin{tikzcd} O \arrow[d, tail, "n"'] \arrow[dd, bend right] \arrow[dr, phantom, "\Downarrow \alpha_j"] & I \arrow[l, "r"'] \arrow[d, "n^*_j"] \arrow[dd, bend left] \\ Y \arrow[d] \arrow[dr, phantom, "\exists! \Downarrow \beta_j"] & X_j \arrow[l] \arrow[d, dashed] \\ Z & X' \arrow[l] \end{tikzcd}
$$

s.th. $\beta_j \circ \alpha_j = \gamma$

$\hat{=}$ "OP-Foliation" Structure!

CONSTRUCTION

$$\forall \quad \begin{tikzcd} O \arrow[d, "n"'] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] & I \\ Y & & \end{tikzcd} \quad : \quad T^{\times}(r,n) := \left\{ \begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "\mathrm{POC}"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d] \arrow[dr, phantom, "\mathrm{PO}"] & I \arrow[d, "n_j"] \\ Y & K_j \arrow[l] \arrow[r] & X_j \end{tikzcd} \right\} \Bigg| \text{ pick one}$$

If the pushout complement exists, it is unique up to iso, as is the PO...

FIBRATIONAL STRUCTURES (II)

- DPO-SEMANTICS:

$$
\forall\ \begin{tikzcd} O & I \arrow[l, "r"'] \arrow[d, "m"] \\ & X \end{tikzcd} : \exists\, S^{*}(r,m) = \left\{ \begin{tikzcd} O \arrow[d] \arrow[r, phantom, "\Downarrow\alpha_j"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y_j & X \arrow[l] \end{tikzcd} \right\}_{j \in S_{r;m}} : \forall \left(\begin{tikzcd} O \arrow[dd] \arrow[rdd, phantom, "\Downarrow\gamma"] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dd, bend left] \\ & X \arrow[d] \\ Y' & X' \arrow[l] \end{tikzcd} \right) : \exists!_{j \in S_{r;m}} : \left(\begin{tikzcd} O \arrow[d] \arrow[dd, bend right] \arrow[r, phantom, "\Downarrow\alpha_j"] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dd, bend left] \\ Y_j \arrow[d, dashed] \arrow[r, phantom, "\Downarrow\exists!\beta_j"] & X \arrow[l] \arrow[d] \\ Y' & X' \arrow[l] \end{tikzcd} \right)
$$

s.th. $\beta_j \circ \alpha_j = \gamma$

"OP-FOLIATION" STRUCTURE!

FIBRATIONAL STRUCTURES (II)

▷ SqPO – SEMANTICS:

$$
\forall \begin{tikzcd} O & I \arrow[l, "r"'] \arrow[d, "m"] \\ & X \end{tikzcd}
\;:\; \exists\, \delta^{\gamma}(r,m) = \begin{tikzcd} O \arrow[d, "m^*"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow \alpha_{r,m}"] \\ Y^* & X \arrow[l, "r^*"'] \end{tikzcd}
\;:\; \forall \begin{tikzcd} O \arrow[ddd] \arrow[ddd, phantom, bend left, "\Downarrow \gamma"] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[ddd, bend left] \\ & X \arrow[dd] \\ & \\ Y' & X' \arrow[l] \end{tikzcd}
\;:\; \begin{tikzcd} O \arrow[d, "m^*"'] \arrow[dd, bend right] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow \alpha_{r,m}"] \arrow[dd, bend left] \\ Y^* \arrow[d, dashed] & X \arrow[l, "r^*"'] \arrow[d] \arrow[dl, phantom, "\Downarrow \exists! \beta"] \\ Y' & X' \arrow[l] \end{tikzcd}
$$

s.th. $\beta \circ \alpha_{r,m} = \gamma$

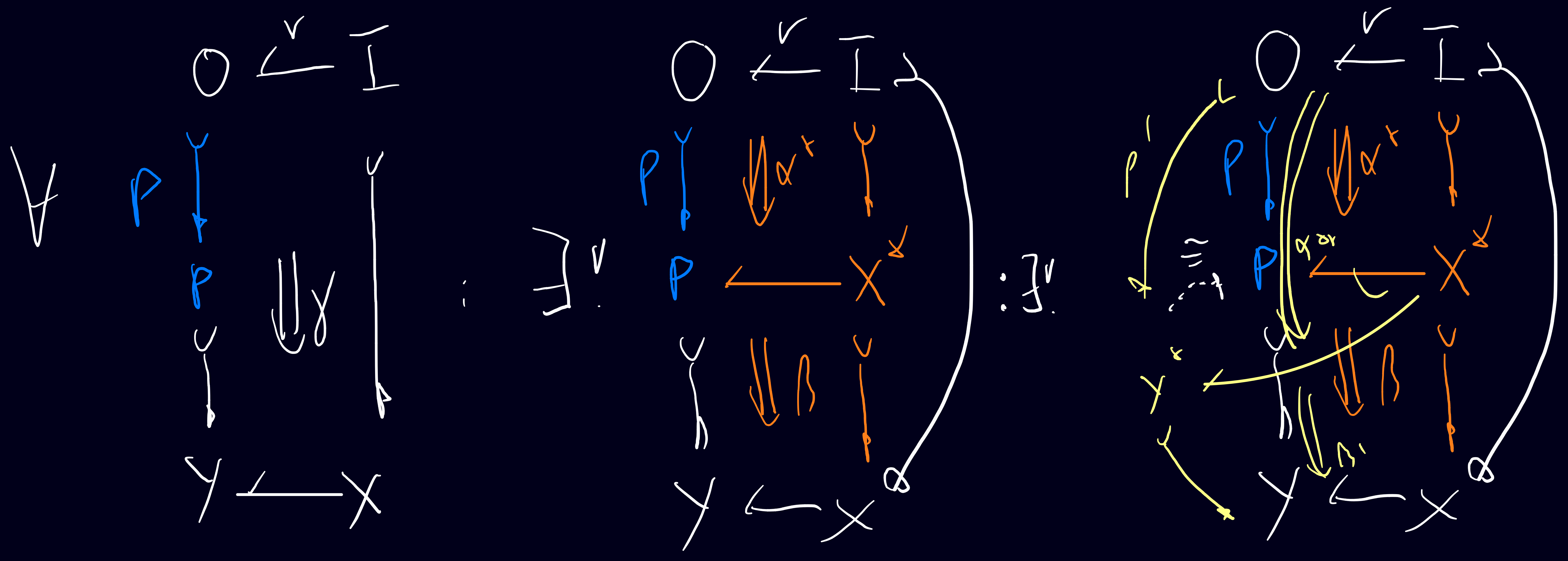
GROTHENDIECK OPFIBRATION STRUCTURE ✓

CONSTRUCTION:

$$\forall \begin{tikzcd} O & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] & I \arrow[d, "m"] \\ & & X \end{tikzcd} : \quad S^{\gamma}(r, m) = \left\{ \begin{tikzcd} O \arrow[d] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d] \arrow[dr, phantom, "FPC"] & I \arrow[d, "m"] \\ Y^{\gamma} \arrow[r] & K^{\gamma} \arrow[r] & X \end{tikzcd} \right\} \Bigg| \text{ pick one}$$

FPC is assumed to always exist

KEY "COMPATIBILITY" PROPERTY



s.th. $\beta \circ \alpha^{\delta} = \gamma$

s.th. $\beta' \circ \alpha^{\delta\delta} = \gamma$

PROOF: obvious for DPO-, not so obvious for SqPO-semantics...

TRANSPORT CONSTRUCTION

$$
\begin{tikzcd}
\mathbb{C}_O \arrow[r, phantom, "\triangleright"] & O \arrow[d, hook, "n_\alpha"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, hook, "m_\alpha"] & \mathrm{Trans}(r, \mathbb{C}_O) \arrow[l, phantom, "\triangleleft"] \\
& Y & X \arrow[l, "r_\alpha"]
\end{tikzcd}
$$

$$n_\alpha \vDash \mathbb{C}_A \iff m_\alpha \vDash \mathrm{Trans}(r, \alpha)$$

- $\mathrm{Trans}(r, \mathrm{true}_O) := \mathrm{true}_I$
- $\mathrm{Trans}(r, \exists(O \xrightarrow{p} P, \mathbb{C}_P)) := \begin{cases} \mathrm{false}_I, & T^\alpha(r,p) = \emptyset \\ \exists(I \xrightarrow{p^*} P^*, \mathrm{Trans}(r^*, \mathbb{C}_P)), & \text{if } T^\alpha(r,p) = \left\{ \begin{tikzcd} O \arrow[d, hook, "p"'] \arrow[dr, phantom, "\Downarrow"] & I \arrow[l, "r"'] \arrow[d, hook, "p^*"] \\ P & P^* \arrow[l] \end{tikzcd} \right\} \end{cases}$
- $\mathrm{Trans}(r, \neg \mathbb{C}_O) := \neg \mathrm{Trans}(r, \mathbb{C}_O)$
- $\mathrm{Trans}(r, \mathbb{C}_O^{(1)} \wedge \mathbb{C}_O^{(2)}) := \mathrm{Trans}(r, \mathbb{C}_O^{(1)}) \wedge \mathrm{Trans}(r, \mathbb{C}_O^{(2)})$

KEY PROPERTIES OF Trans

(I) $\mathrm{Trans}(X \xleftarrow{id_X} X, \mathbb{C}_X) \equiv \mathbb{C}_X$

(II) $\mathrm{Trans}(Y \xleftarrow{r} X, \mathrm{Trans}(Z \xleftarrow{s} Y, \mathbb{C}_Z)) \equiv \mathrm{Trans}(s \circ r, \mathbb{C}_Z)$

COMPATIBILITY OF shift and Trans

$$\mathbb{C}_O \vartriangleright \begin{tikzcd} O \arrow[d, "n_\alpha"'] & I \arrow[l, "r"'] \arrow[d, "m_\alpha"] \arrow[l, phantom, shift right=6, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd}$$

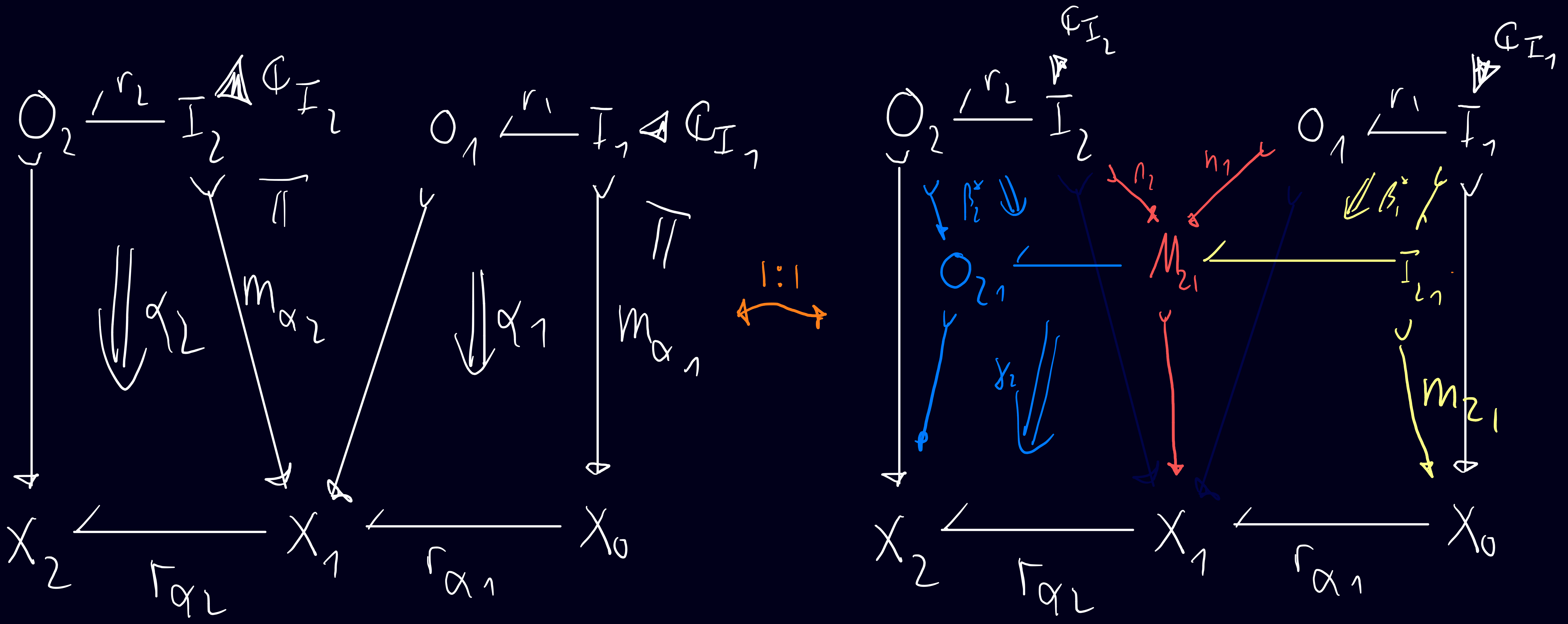
$$\mathrm{Shift}(m_\alpha, \mathrm{Trans}(r, \mathbb{C}_O))$$

$$\equiv \mathrm{Trans}(s, \mathrm{Shift}(n_\alpha, \mathbb{C}_O))$$

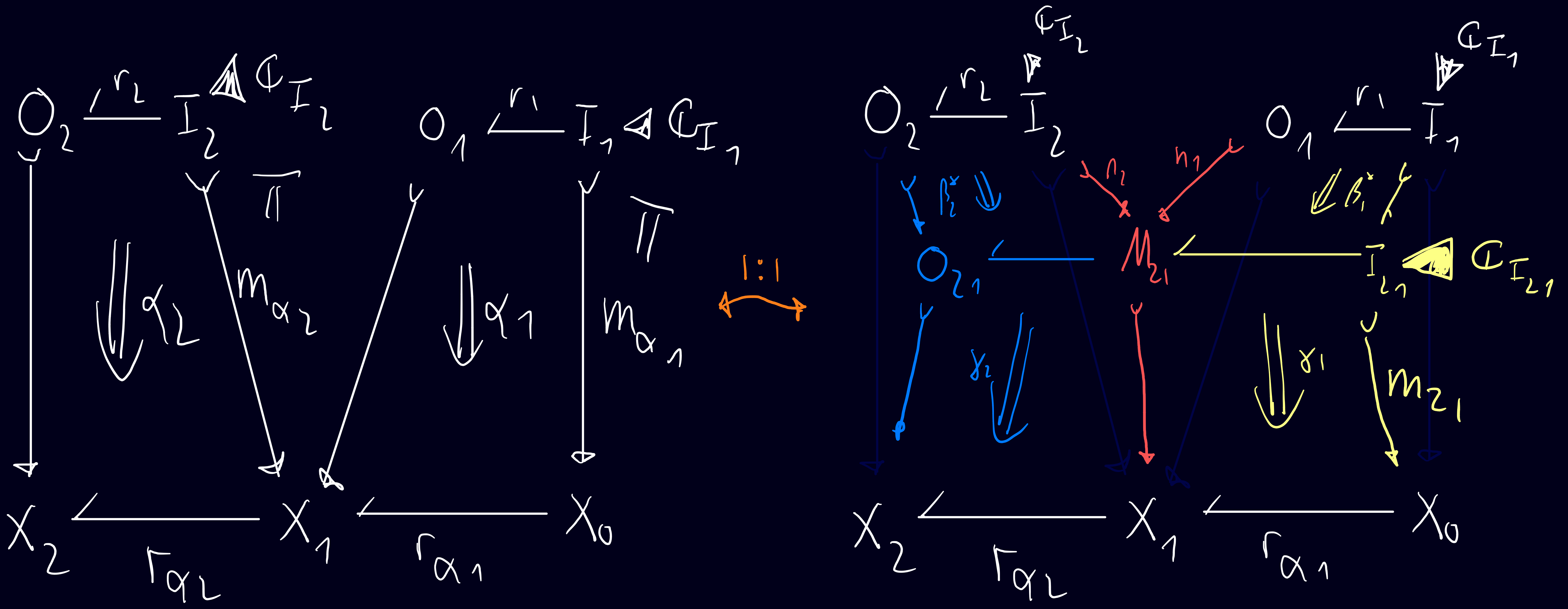
Concurrency theorem

$$
\begin{tikzcd}
O_2 \arrow[d] & I_2 \triangleleft \mathsf{c}_{I_2} \arrow[l, "r_2"'] \arrow[dr, "m_{\alpha_2}"] \arrow[dl, phantom, "\Downarrow\alpha_2"] & & O_1 \arrow[dl] & I_1 \triangleleft \mathsf{c}_{I_1} \arrow[l, "r_1"'] \arrow[d, "m_{\alpha_1}"] \arrow[dl, phantom, "\Downarrow\alpha_1"] \\
X_2 & & X_1 \arrow[ll, "r_{\alpha_2}"'] & & X_0 \arrow[ll, "r_{\alpha_1}"']
\end{tikzcd}
\quad \overset{1:1}{\longleftrightarrow} \quad
\begin{tikzcd}
& \mathsf{c}_{I_2} \arrow[d, phantom, "\triangledown"] & & & \mathsf{c}_{I_1} \arrow[d, phantom, "\triangledown"] \\
O_2 \arrow[ddd] & I_2 \arrow[l, "r_2"'] \arrow[dddr] \arrow[dr, "n_2"] & & O_1 \arrow[dddl] \arrow[dl, "n_1"'] & I_1 \arrow[l, "r_1"'] \arrow[ddd] \\
& & M_{21} \arrow[dd] & & \\
& & & & \\
X_2 & & X_1 \arrow[ll, "r_{\alpha_2}"'] & & X_0 \arrow[ll, "r_{\alpha_1}"']
\end{tikzcd}
$$

CONCURRENCY THEOREM

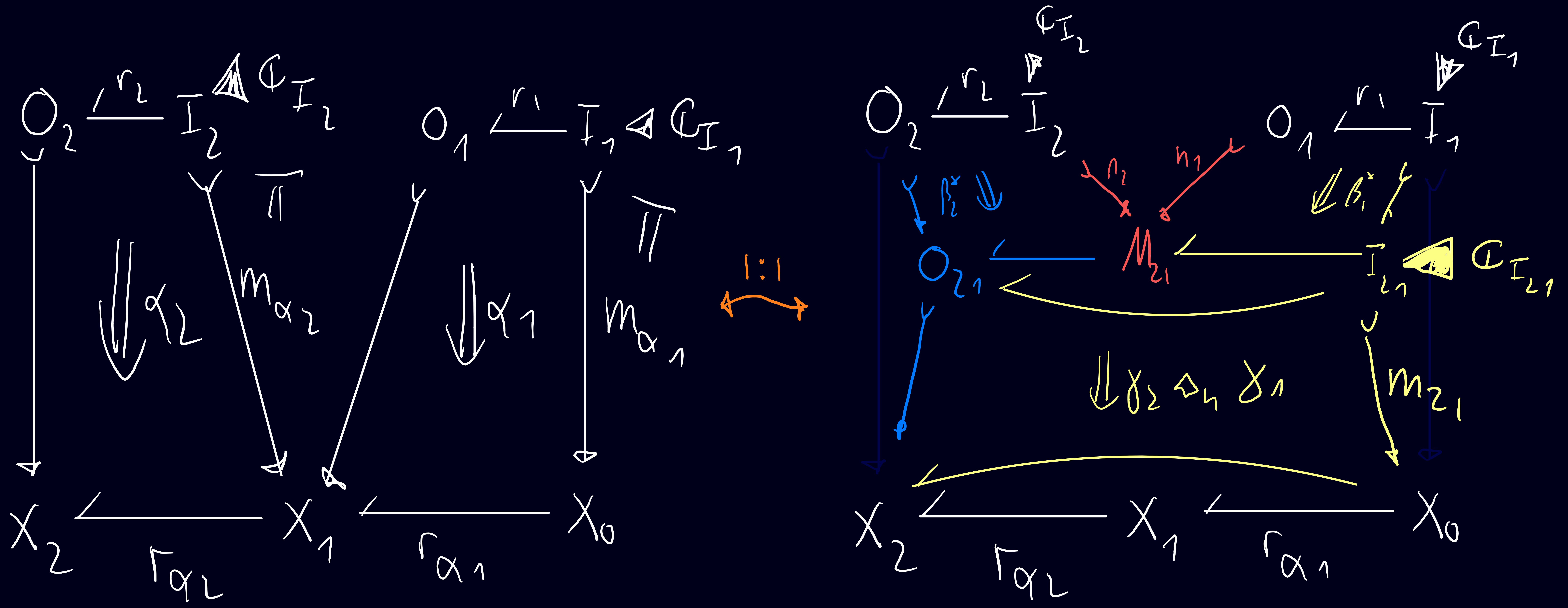


Concurrency Theorem



$$c_{I_{21}} = \mathrm{Trans}\left(M_{21} \leftarrow I_{21},\ \mathrm{Shift}(n_2, c_{I_2})\right) \wedge \mathrm{Shift}(n_1, c_{I_1})$$

CONCURRENCY THEOREM

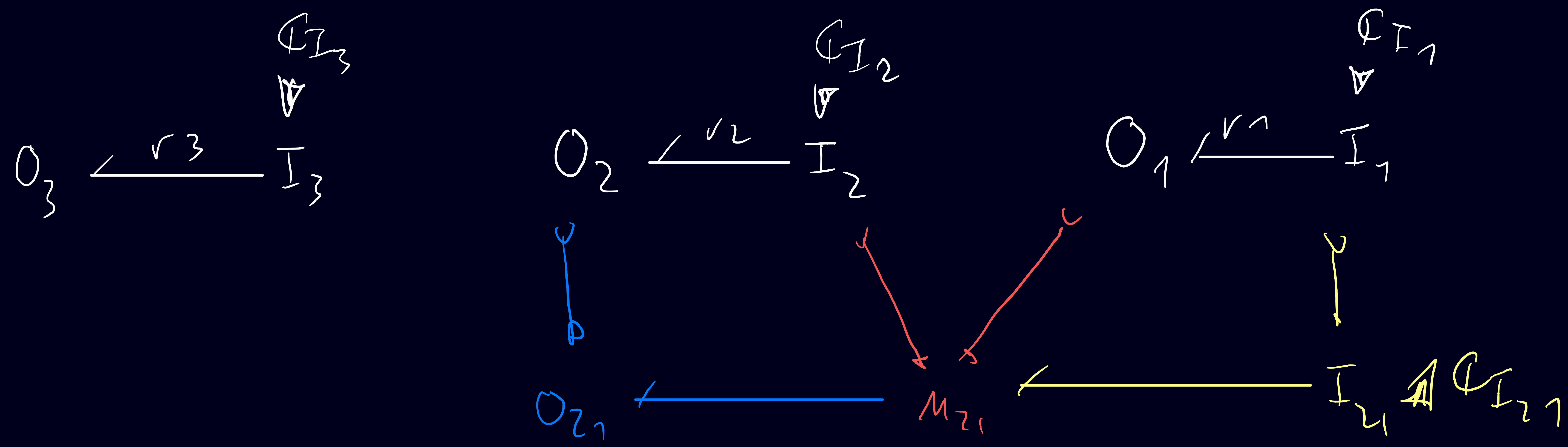


$$c_{I_{21}} = \mathrm{Trans}(M_{21} \leftarrow I_{21}, \mathrm{Shift}(n_2, c_{I_2})) \wedge \mathrm{Shift}(n_1, c_{I_1})$$

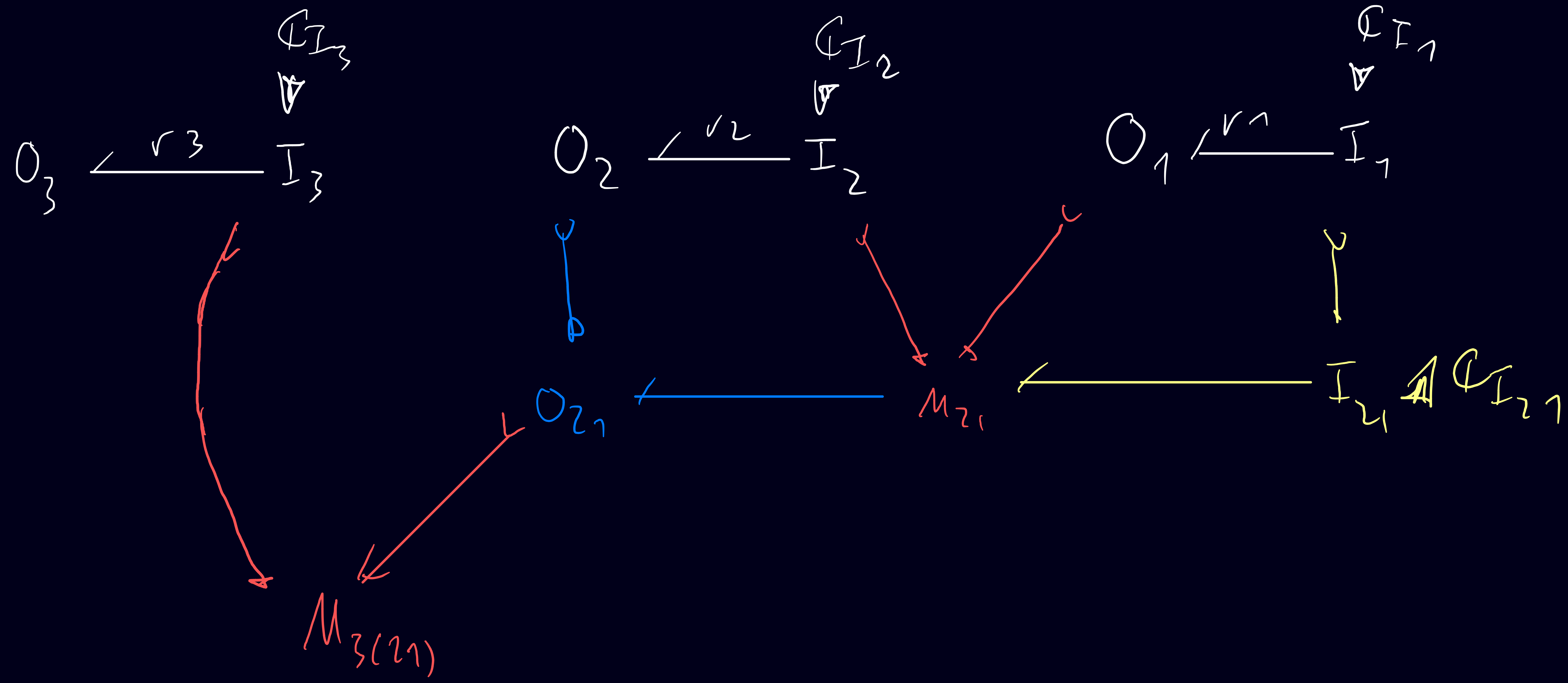
Associativity Theorem

$$
\begin{tikzcd}
 & \mathsf{c}_{I_3} \arrow[d, phantom, "\triangledown"] & & \mathsf{c}_{I_2} \arrow[d, phantom, "\triangledown"] & & & \mathsf{c}_{I_1} \arrow[d, phantom, "\triangledown"] \\
O_3 & I_3 \arrow[l, "r_3"'] & O_2 & I_2 \arrow[l, "r_2"'] \arrow[dr, hook] & & O_1 \arrow[dl, hook'] & I_1 \arrow[l, "r_1"'] \\
 & & & & M_{21} & &
\end{tikzcd}
$$

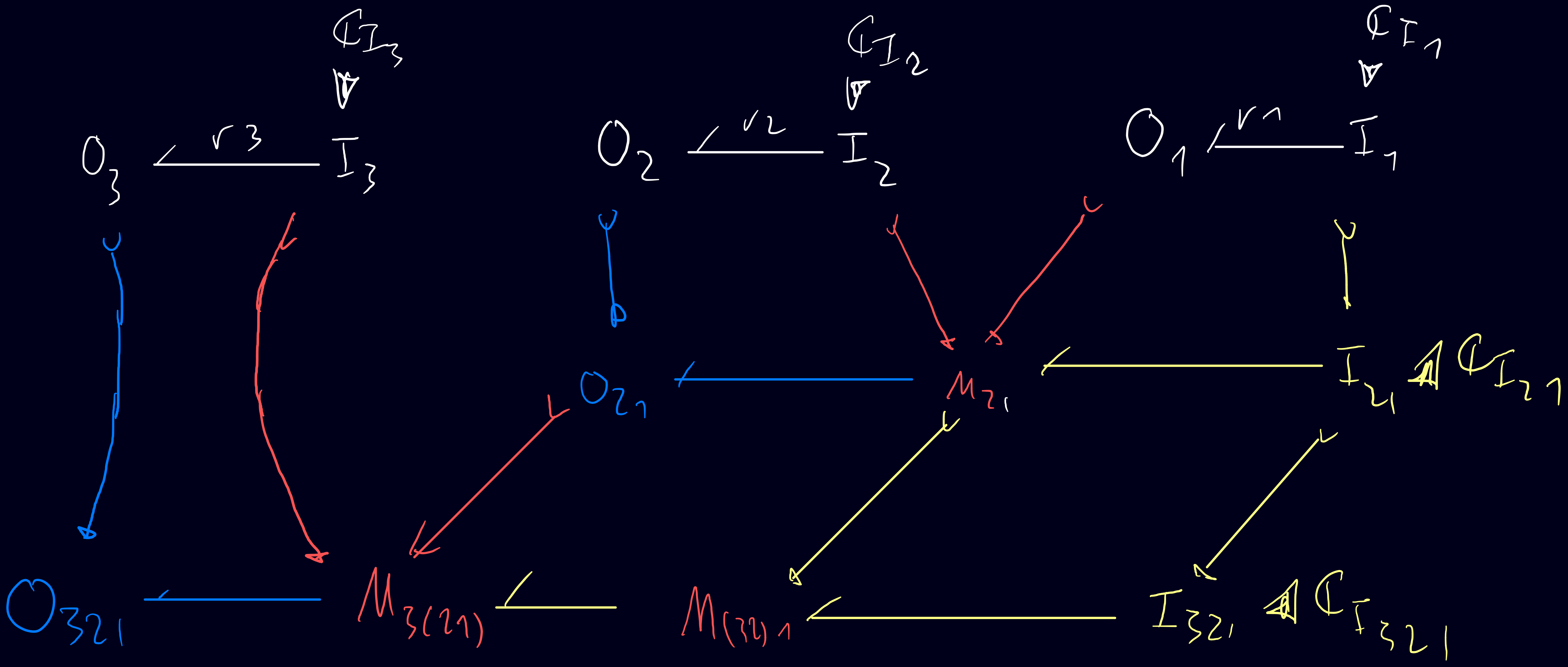
ASSOCIATIVITY THEOREM



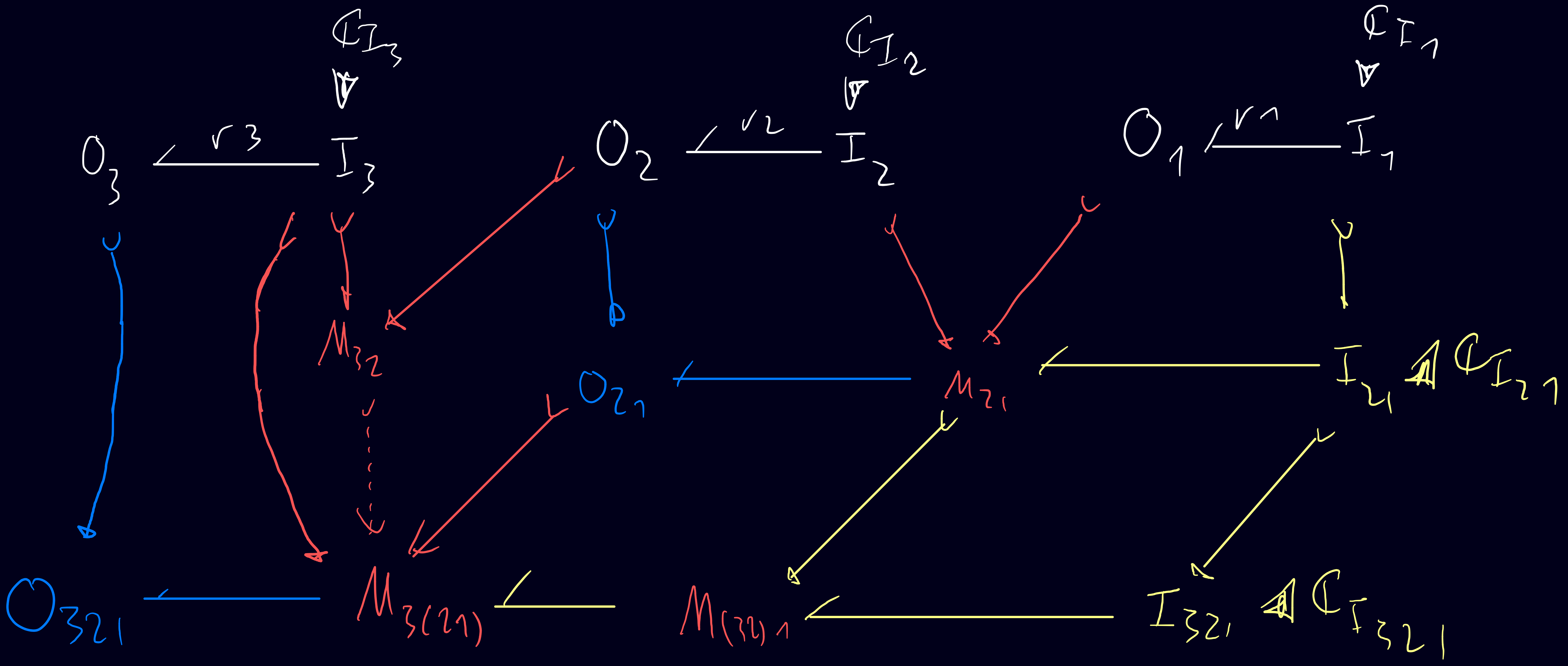
ASSOCIATIVITY THEOREM



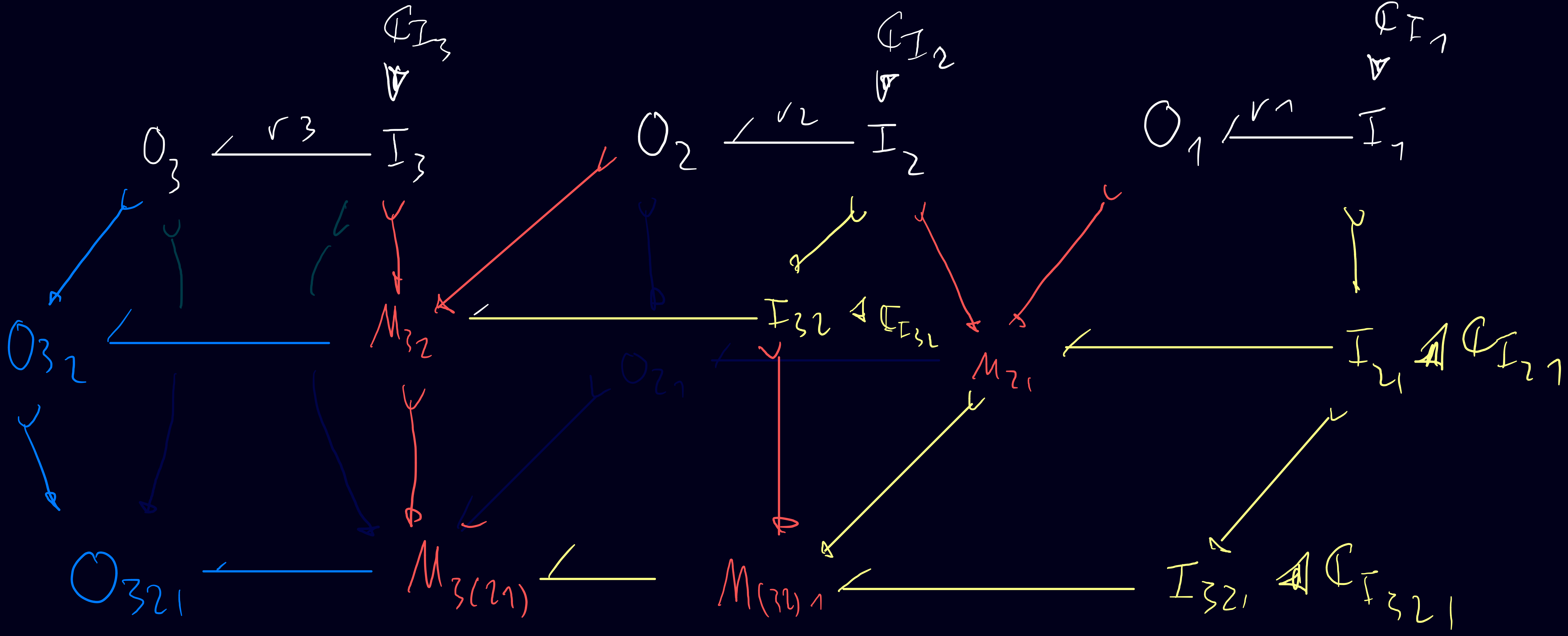
Associativity Theorem



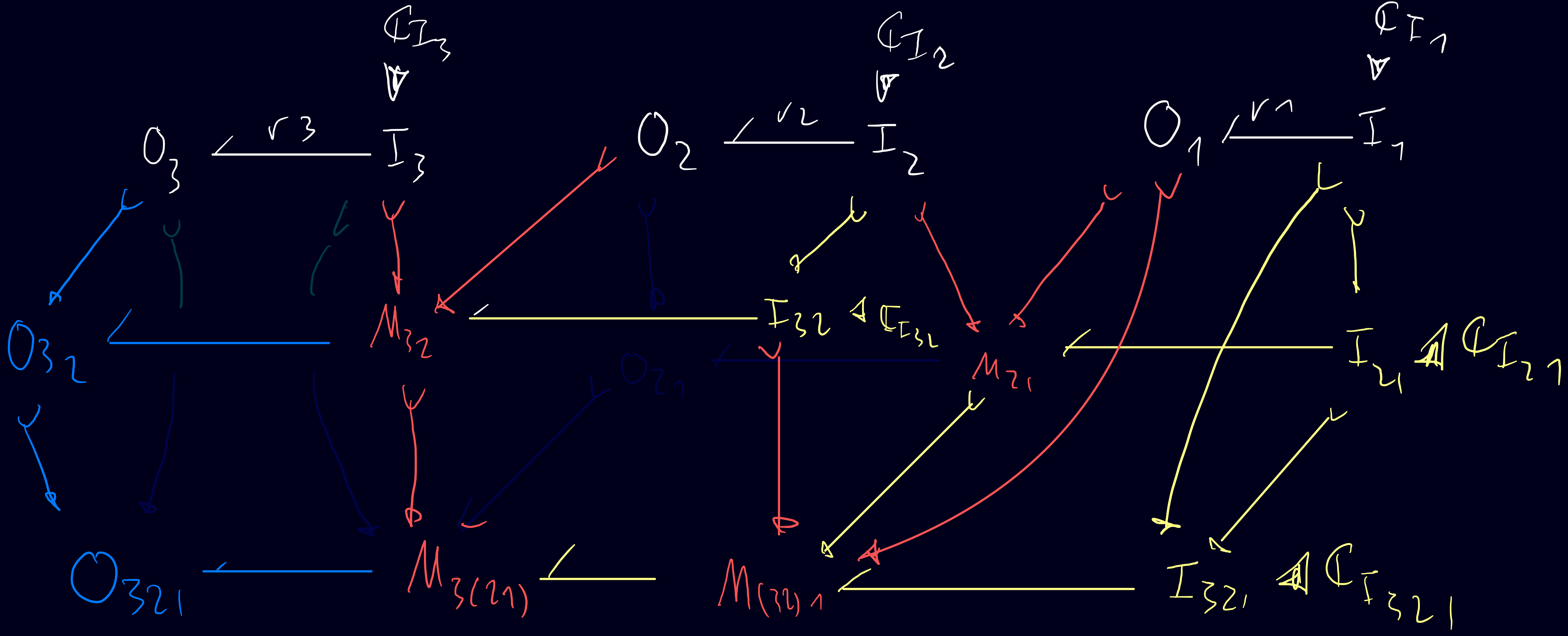
ASSOCIATIVITY THEOREM



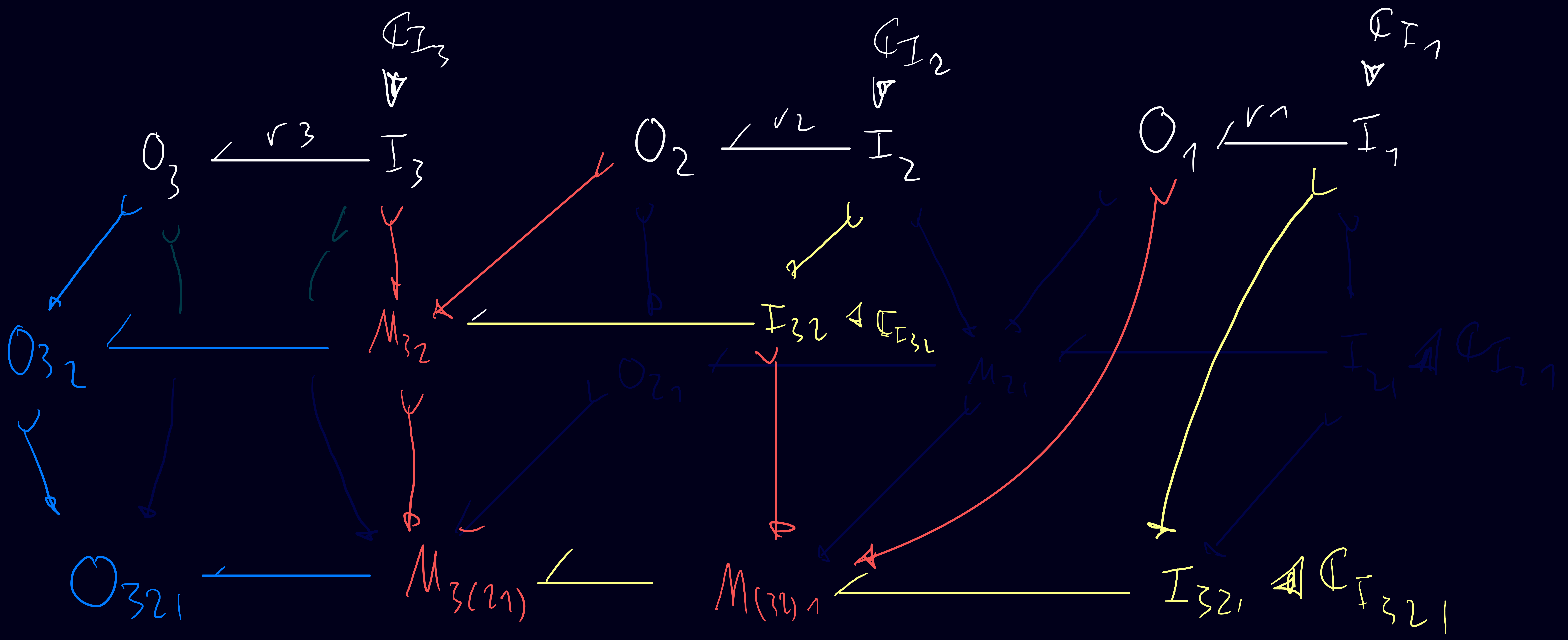
Associativity Theorem



ASSOCIATIVITY THEOREM



ASSOCIATIVITY THEOREM



III. Refinements

Conditional Reactive Systems*

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FSTTCS 2011

Categories of Conditions:

$\forall X \in obj(\mathcal{C}):$

- $obj(\mathcal{C}_X) := \{\mathbb{C}_X\}$
- $mor(\mathcal{C}_X) := \{\mathbb{C}_X^{(1)} \models \mathbb{C}_X^{(2)}\}$

$\forall A \xrightarrow{\varphi} B \in \mathcal{C}:$

$$\begin{tikzcd}
\mathcal{C}_A \arrow[rr, "Shift(\varphi{,}\, -)"] \arrow[rr, phantom, bend left=25, "\top"] \arrow[rr, phantom, bend right=25, "\top"] & & \mathcal{C}_B \arrow[ll, bend right=50, "\forall\varphi"'] \arrow[ll, bend left=50, "\exists\varphi"]
\end{tikzcd}$$

III. REFINEMENTS

CONSTRAINT-PRESERVING CONDITIONS

[Habel & Pennemann]

+ COMPOSITIONALITY

Nicolas Behr (2021). On Stochastic Rewriting and Combinatorics via Rule-Algebraic Methods. Invited Paper in Patrick Bahr (ed.): Proceedings 11th International Workshop on Computing with Terms and Graphs (TERMGRAPH 2020), Online, 5th July 2020, Electronic Proceedings in Theoretical Computer Science 334, pp. 11–28..

Nicolas Behr, Jean Krivine, Jakob L. Andersen, Daniel Merkle (2021). Rewriting theory for the life sciences: A unifying theory of CTMC semantics. In: Theoretical Computer Science.