Operational Methods in the Study of Sobolev-Jacobi Polynomials

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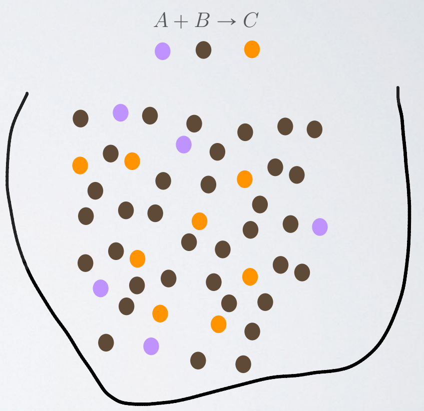
Delbrück's insight: probability generating function evolution equation for chemical reaction systems



- **pure state:** characterized by **numbers of each type of particle**

$$|\underline{n}\rangle = |n_A, n_B, n_C\rangle$$

- **probability generating function:** given a time-dependent probability distribution $|\Psi\rangle(t) = \sum_{\underline{n}\geqslant 0} p_{\underline{n}}(t)\,|\underline{n}\rangle$ over pure states,

$$|\psi\rangle \quad \leftrightarrow \quad P(t;\underline{x}) := \sum_{\underline{n}\geqslant 0} p_{\underline{n}}(t)\underline{x}^{\underline{n}} \tag{1}$$

with $\underline{x}^{\underline{n}} = x_A^{n_A} x_B^{n_B} x_C^{n_C}$

Delbrück [1]

The **master equation** for a chemical reaction system with reactions

$$\underline{i} \cdot \underline{X} \xrightarrow{r_{\underline{i},\underline{o}}} \underline{o} \cdot \underline{X}$$

reads (with $\hat{\underline{x}}[\underline{x}^{\underline{n}}] = \underline{x}^{\underline{n}+\underline{1}}$)

$$\frac{\partial}{\partial t} P(t;\underline{x}) = \sum_{\underline{i},\underline{o}} r_{\underline{i},\underline{o}} \left((\hat{\underline{x}})^{\underline{o}} - (\hat{\underline{x}})^{\underline{i}} \right) \left(\frac{\partial}{\partial \underline{x}} \right)^{\underline{i}} P(t;\underline{x}) \tag{2}$$

[1] Max Delbrück. "Statistical fluctuations in autocatalytic reactions". In: *The Journal of Chemical Physics* 8.1 (1940), pp. 120–124

Why Sobolev-Jacobi polynomials?

- **Jacobi differential operator:** for $\alpha, \beta \in \mathbb{R}$,

$$D_{Jac}^{(\alpha,\beta)} := (1-x^2)\partial_x^2 + ((\beta-\alpha) - (\alpha+\beta+2)x)\partial_x \tag{3}$$

- **For $\alpha, \beta > -1$:** polynomial eigenfunctions are the **Jacobi polynomials**,

$$\begin{aligned} D_{Jac}^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(x) &= -n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) \\ P_n^{(\alpha,\beta)}(x) &= \sum_{k=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-k}\left(\frac{x+1}{2}\right)^k\left(\frac{x-1}{2}\right)^{n-k}, \end{aligned} \tag{4}$$

which are **orthogonal** w.r.t. the **inner product** $\Phi^{(\alpha,\beta)}$ defined via

$$\forall p,q \in \mathbb{R}[x]: \quad \Phi^{(\alpha,\beta)}(p,q) := \int_{-1}^{+1} dx\,(1-x)^{\alpha}(1+x)^{\beta}p(x)q(x)\,, \tag{5}$$

such that more explicitly (with $\varphi_n^{(\alpha,\beta)} \in \mathbb{R}$ some non-zero constants)

$$\forall m,n \in \mathbb{Z}_{\geqslant 0}: \ \Phi^{(\alpha,\beta)}(P_m^{(\alpha,\beta)}, P_n^{(\alpha,\beta)}) = \delta_{m,n}\varphi_n^{(\alpha,\beta)}\,, \qquad \forall n \in \mathbb{Z}_{\geqslant 0}: \ P_n^{(\alpha,\beta)}(x) \in L^2_{\Phi^{(\alpha,\beta)}}([-1,1]) \tag{6}$$

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- **But:** e.g. for **chemical reaction** $2A \rightharpoonup \varnothing$, the evolution operator is $D_{Jac}^{(-1,-1)} = (1-x^2)\partial_x^2$

$\Rightarrow$ **Problems:**

$$P_0^{(-1,-1)}(x) \notin L^2_{\Phi^{(\alpha,\beta)}}([-1,1]) \quad \text{and} \quad P_1^{(-1,-1)}(x) = 0 \qquad \Rightarrow \textbf{no OPS!} \tag{7}$$

Solution: Sobolev-Jacobi polynomials

- **Kwon & Littlejohn (1990's)** [2]: resolve the problem for $\alpha \leqslant -1$ and/or $\beta \leqslant -1$ via replacing $\Phi^{(\alpha,\beta)}$ with the **Sobolev inner products** $\Phi_{A,B}^{(-1,-1)}$ and $\Phi_C^{(-1,\beta)}$ (for $\beta > -1$) defined as follows:

$$\forall p,q \in \mathbb{R}[x]: \quad \Phi_{A,B}^{(-1,-1)}(p,q) := Ap(1)q(1) + Bp(-1)q(-1) + \int_{-1}^{+1} dx\, p'(x)q'(x)$$
$$\Phi_C^{(-1,\beta)}(p,q) := Cp(1)q(1) + \int_{-1}^{+1} dx\, (x+1)^{\beta+1} p'(x)q'(x) \qquad (\beta > -1) \tag{8}$$

here, $p'(x)$ and $q'(x)$ denote the **first derivatives of the polynomials**, and $A,B,C \in \mathbb{R}$ are parameters. For the resulting inner products to be **positive definite**, the parameters $A,B,C \in \mathbb{R}$ must satisfy the following conditions: for $\Phi_{A,B}^{(-1,-1)}$, A, and B must verify

$$A+B>0, \quad A(\gamma+1)^2 + B(\gamma-1)^2 + 2 \neq 0, \quad A(\gamma+1) + B(\gamma-1) = 0, \quad \gamma := (B-A)/(A+B) \tag{9}$$

while for $\Phi_C^{(-1,\beta)}$ the parameter C must verify

$$C > 0 \tag{10}$$

[2] Kil H. Kwon and L.L. Littlejohn. "Sobolev orthogonal polynomials and second-order differential equations". In: *The Rocky Mountain journal of mathematics* (1998), pp. 547–594

Solution: Sobolev-Jacobi polynomials

- The **Sobolev-Jacobi orthogonal polynomials** $\tilde{P}_n^{(\alpha,\beta)}(x)$ are defined for $\alpha = -1,\ \beta = -1$ as

$$\begin{aligned}
\tilde{P}_0^{(-1,-1)}(x) &:= 1 \\
\tilde{P}_1^{(-1,-1)}(x) &:= x + \gamma \qquad (\gamma := (B-A)/(A+B)) \\
\tilde{P}_{n\geqslant 2}^{(-1,-1)}(x) &:= \binom{2n-2}{n}^{-1} \sum_{k=1}^{n-1} \binom{n-1}{k} \binom{n-1}{n-k} (x-1)^{n-k}(x+1)^k
\end{aligned} \tag{8}$$

where $\gamma = (B-A)/(A+B)$, while for the parameters $\alpha = -1,\ \beta > -1$ one defines

$$\begin{aligned}
\tilde{P}_0^{(-1,\beta)}(x) &:= 1 \\
\tilde{P}_1^{(-1,\beta)}(x) &:= x - 1 \\
\tilde{P}_{n\geqslant 2}^{(-1,\beta)}(x) &:= \binom{2n+\beta-1}{n}^{-1} \sum_{k=0}^{n} \binom{n-1}{k} \binom{n+\beta}{n-k} (x-1)^{n-k}(x+1)^k
\end{aligned} \tag{9}$$

Propositions 4.2 and 4.3 of [2]

The polynomials $\tilde{P}_n^{(\alpha,\beta)}(x)$ form a **complete orthogonal system of polynomial eigenfunctions** of the Jacobi differential operator at parameters in the aforementioned parameter ranges.

Plan

I An interesting technique by Gurappa and Panigrahi

Recasting the Sobolev-Jacobi polynomials in a novel operational form.

II Reformulating multi-variate umbral calculus via integral operators

From generalized polynomials to a calculus for special functions and formal power series.

III Operational method results on Sobolev-Jacobi polynomials

Discoveries and results on algebraic structure, connection coefficients and all-order higher-order lacunary exponential generating functions of the SJ polynomials.

Paper: **Nicolas Behr, Giuseppe Dattoli, Gérard H. E. Duchamp, Silvia Licciardi, and Karol A. Penson. "Operational Methods in the Study of Sobolev-Jacobi Polynomials".** In: *Mathematics* 7.2 (2019)

An innovative technique developed by Gurappa and Panigrahi

An innovative technique to determine polynomial eigenfunctions

The technique of Gurappa and Panigrahi [3]

For a **differential operator** $\widehat{\mathbb{D}}$ in the variable x, suppose there exist **polynomial eigenfunctions** $q_n(x) \in \mathbb{C}[x]$ for each $n \geqslant 0$ such that $\widehat{\mathbb{D}}\, q_n(x) = \lambda_n\, q_n(x) \quad (\lambda_n \in \mathbb{C})$. To calculate the $q_n(x)$'s,

[3] N. Gurappa, C. Nagaraja Kumar, and P. K. Panigrahi. "New exactly and conditionally exactly solvable N-body problems in one dimension". In: *Modern Physics Letters A* 11.21 (July 1996), pp. 1737–1744; N. Gurappa and P. K. Panigrahi. "Equivalence of the Sutherland model to free particles on a circle". In: *arXiv preprint hep-th/9908127* (1999); N. Gurappa, P. K. Panigrahi, T. Shreecharan, and S. Sree Ranjani. "A Novel Method to Solve Familiar Differential Equations and its Applications". In: *Frontiers of Fundamental Physics 4*. Springer US, 2001, pp. 269–277; N. Gurappa, P. K. Panigrahi, and T Shreecharan. "Linear differential equations and orthogonal polynomials: A novel approach". In: *arXiv preprint math-ph/0203015* (2002); T. Shreecharan, P. K. Panigrahi, and J. Banerji. "Coherent states for exactly solvable potentials". In: *Physical Review A* 69.1 (Jan. 2004); N. Gurappa and P. K. Panigrahi. "On polynomial solutions of the Heun equation". In: *Journal of Physics A: Mathematical and General* 37.46 (Nov. 2004), pp. L605–L608

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(i) **decompose** the operator $\widehat{\mathbb{D}} - \lambda_n$ as follows (with $\hat{x}[x^n] := x^{n+1}$ and $\hat{D}_x := \hat{x}\partial_x$):

$$\widehat{\mathbb{D}} - \lambda_n = \underbrace{F_n(\hat{D_x})}_{\text{“diagonal part”}} + \underbrace{N(\hat{x}, \partial_x)}_{\text{“non-diagonal part”}} , \qquad F_n(\hat{D_x})) = \sum_{k \geqslant 0} f_{n,k}\, \hat{D}_x^k , \quad N(\hat{x}, \partial_x) = \sum_{\substack{k,\ell \geqslant 0 \\ k \neq \ell}} g_{n,k,\ell}\, \hat{x}^k \partial_x^\ell \tag{10}$$

[3] N. Gurappa, C. Nagaraja Kumar, and P. K. Panigrahi. “New exactly and conditionally exactly solvable N-body problems in one dimension”. In: *Modern Physics Letters A* 11.21 (July 1996), pp. 1737–1744; N. Gurappa and P. K. Panigrahi. “Equivalence of the Sutherland model to free particles on a circle”. In: *arXiv preprint hep-th/9908127* (1999); N. Gurappa, P. K. Panigrahi, T. Shreecharan, and S. Sree Ranjani. “A Novel Method to Solve Familiar Differential Equations and its Applications”. In: *Frontiers of Fundamental Physics 4*. Springer US, 2001, pp. 269–277; N. Gurappa, P. K. Panigrahi, and T Shreecharan. “Linear differential equations and orthogonal polynomials: A novel approach”. In: *arXiv preprint math-ph/0203015* (2002); T. Shreecharan, P. K. Panigrahi, and J. Banerji. “Coherent states for exactly solvable potentials”. In: *Physical Review A* 69.1 (Jan. 2004); N. Gurappa and P. K. Panigrahi. “On polynomial solutions of the Heun equation”. In: *Journal of Physics A: Mathematical and General* 37.46 (Nov. 2004), pp. L605–L608

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$$\widehat{\mathbb{D}} - \lambda_n = \underbrace{F_n(\hat{D_x})}_{\text{"diagonal part"}} + \underbrace{N(\hat{x}, \partial_x)}_{\text{"non-diagonal part"}} \;, \qquad F_n(\hat{D_x})) = \sum_{k\geqslant 0} f_{n,k}\, \hat{D}_x^k \,, \quad N(\hat{x}, \partial_x) = \sum_{\substack{k,\ell\geqslant 0 \\ k\neq\ell}} g_{n,k,\ell}\, \hat{x}^k \partial_x^\ell \tag{10}$$

(ii) If $F_n(\hat{D}_x)x^n = 0$, one finds (for some normalization constants $c_n \in \mathbb{C}$

$$q_n(x) = c_n \frac{1}{1 + \frac{1}{F_n(\hat{D}_x)} N(\hat{x}, \partial_x)} x^n = c_n \sum_{m\geqslant 0} (-1)^m \left[\frac{1}{F_n(\hat{D}_x)} N(\hat{x}, \partial_x)\right]^{\circ m} x^n \,. \tag{11}$$

[3] N. Gurappa, C. Nagaraja Kumar, and P. K. Panigrahi. "New exactly and conditionally exactly solvable N-body problems in one dimension". In: *Modern Physics Letters A* 11.21 (July 1996), pp. 1737–1744; N. Gurappa and P. K. Panigrahi. "Equivalence of the Sutherland model to free particles on a circle". In: *arXiv preprint hep-th/9908127* (1999); N. Gurappa, P. K. Panigrahi, T. Shreecharan, and S. Sree Ranjani. "A Novel Method to Solve Familiar Differential Equations and its Applications". In: *Frontiers of Fundamental Physics 4.* Springer US, 2001, pp. 269–277; N. Gurappa, P. K. Panigrahi, and T Shreecharan. "Linear differential equations and orthogonal polynomials: A novel approach". In: *arXiv preprint math-ph/0203015* (2002); T. Shreecharan, P. K. Panigrahi, and J. Banerji. "Coherent states for exactly solvable potentials". In: *Physical Review A* 69.1 (Jan. 2004); N. Gurappa and P. K. Panigrahi. "On polynomial solutions of the Heun equation". In: *Journal of Physics A: Mathematical and General* 37.46 (Nov. 2004), pp. L605–L608

Specializing the technique to the case of SJ polynomials

Given the **Jacobi differential operator**

$$D_{Jac}^{(\alpha,\beta)} = (1-x^2)\partial_x^2 + ((\beta-\alpha)-(\alpha+\beta+2)x)\partial_x \overset{!}{=} F_n^{(\alpha,\beta)}(\hat{D}_x) + N^{(\alpha,\beta)}(\hat{x},\partial_x)\,,$$

based upon the **auxiliary identity** $\hat{D}_x^2 = \hat{x}^2\partial_x^2 + \hat{D}_x$ one may compute concretely

$$\begin{aligned} F_n^{(\alpha,\beta)}(\hat{D}_x) &:= -(\hat{D}_x - n)(\hat{D}_x + n + \alpha + \beta + 1) \\ N^{(\alpha,\beta)}(\hat{x},\partial_x) &= \partial_x^2 + (\beta-\alpha)\partial_x \end{aligned} \tag{12}$$

and thus obtains the following variant of the **Sobolev-Jacobi polynomials**:

$$\begin{aligned} \tilde{P}_n^{(\alpha,\beta)}(x) &= c_n \frac{1}{1+\frac{1}{\hat{F}_n}\hat{N}} x^n = c_n \sum_{m\geqslant 0} (-1)^m \left[\tfrac{1}{\hat{F}_n}\hat{N}\right]^m x^n \\ &= c_n \sum_{m\geqslant 0} \left[\tfrac{1}{(\hat{D}_x-n)(\hat{D}_x+n+\alpha+\beta+1)} \left(\partial_x^2 + (\beta-\alpha)\partial_x\right)\right]^m x^n\,. \end{aligned} \tag{13}$$

Exponential generating functions

Case $\alpha=\beta=-1$ of the variant of the **Sobolev-Jacobi polynomials** (and with $c_n:=1$ for all $n\geqslant 0$):

$$\tilde{P}_n^{(-1,-1)}(x)=\sum_{m\geqslant 0}\left[\frac{1}{(\hat{D}_x-n)(\hat{D}_x+n-1)}\right]^m x^n\,. \tag{14}$$

Lemma

For any **entire function** $f\equiv f(\hat{D}_x)$ in the operator $\hat{D}_x$, and for any $p,q\in\mathbb{Z}_{\geqslant 0}$, it holds that

$$f(\hat{D}_x)\hat{x}^p\,\partial_x^q=\hat{x}^p\,\partial_x^q f(\hat{D}_x+p-q)\,,\quad \hat{x}^p\,\partial_x^q f(\hat{D}_x)=f(\hat{D}_x-p+q)\hat{x}^p\,\partial_x^q \tag{15}$$

Proposition: exponential formula for the monic SJ polynomials $\tilde{P}_n^{(-1,-1)}(x)$

$$\tilde{P}_n^{(-1,-1)}(x)=e^{-\frac{1}{2}\frac{1}{\hat{D}_x+n-1}\partial_x^2}x^n \tag{16}$$

Proof: Starting from the original definition of $\tilde{P}_n^{(-1,-1)}(x)$, the claim follows from the following application of the above Lemma:

$$\left[\frac{1}{(\hat{D}_x-n)(\hat{D}_x+n-1)}\partial_x^2\right]^m x^n=\left[\frac{1}{(\hat{D}_x+n-1)}\partial_x^2\right]^m\left(\prod_{j=1}^{m}\frac{1}{\hat{D}_x-n-2j}\right)x^n=\frac{1}{m!}\left(-\frac{1}{2}\right)^m\left[\frac{1}{(\hat{D}_x+n-1)}\partial_x^2\right]^m x^n$$

Exponential generating functions

- It will prove convenient to introduce the following **bi-variate polynomials** $P_n(x,y)$:

$$P_n(x,y) := e^{\mathbb{B}}(xy)^n \tag{17}$$

where the **differential operator** $\mathbb{B}$ is defined as

$$\mathbb{B} := \mathbf{b}_0\,\partial_x^2\,, \quad \mathbf{b}_p := -\frac{1}{2}\frac{1}{\hat{D}_x+\hat{D}_y+p-1} \quad (p \in \mathbb{Z}_{\geqslant 0}) \tag{18}$$

$\Rightarrow$ this allows us to re-express the polynomials $\tilde{P}_n^{(-1,-1)}(x)$ in the form $\tilde{P}_n^{(-1,-1)}(x) = (P_n(x,y))\,|_{y\to 1}$

Key result: exponential generating function (EGF) for $P_n(x,y)$ polynomials [4]

The **EGF** $\mathcal{G}(\lambda;x,y)$ of the polynomials $P_n(x,y)$, defined as

$$\mathcal{G}(\lambda;x,y) := \sum_{n\geqslant 0}\frac{\lambda^n}{n!}P_n(x,y) = e^{\mathbb{B}}e^{\lambda xy} \tag{19}$$

takes the explicit form

$$\mathcal{G}(\lambda;x,y) = \sum_{n\geqslant 0}\frac{(\lambda xy)^n}{n!}\sum_{m\geqslant 0}\frac{1}{m!}\left(-\frac{\lambda^2y^2}{4}\right)^m\frac{\Gamma(m+n-\frac{1}{2})}{\Gamma(2m+n-\frac{1}{2})} = \sum_{n\geqslant 0}\frac{(\lambda xy)^n}{n!}\,{}_1F_2\left[\begin{matrix} n-\frac{1}{2} \\ \frac{n}{2}-\frac{1}{4},\ \frac{n}{2}+\frac{1}{4}\end{matrix};-\frac{\lambda^2y^2}{16}\right]. \tag{20}$$

[4] Nicolas Behr, Giuseppe Dattoli, Gérard H. E. Duchamp, Silvia Licciardi, and Karol A. Penson. “Operational Methods in the Study of Sobolev-Jacobi Polynomials”. In: *Mathematics* 7.2 (2019)

Reformulation of multi-variate umbral calculus in terms of integral operators

Two integrals of interest

- **gamma function**:

$$\Gamma(z) := \int_0^\infty dt\, e^{-t} t^{z-1} \tag{21}$$

It is well-known that $\Gamma(z)$ thus defined (and for $Re(z) < 0$ by analytic continuation) is a **meromorphic function** with **no zeros**, and **simple poles** at $z = -n$ ($n \in \mathbb{Z}_{\leqslant 0}$) of residue $(-1)^n/n!$.

- **reciprocal gamma function**:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_\gamma dt\, e^t t^{-z} . \tag{22}$$

Here, t^{-z} is understood as having its principal value where t crosses the positive real axis and is continuous, and the integration is taken along the **Hankel contour** γ. The reciprocal gamma function $1/\Gamma(z)$ is an **entire function** with **simple zeros** at $z = -n$ ($n \in \mathbb{Z}_{\leqslant 0}$).

- The gamma function (resp. reciprocal gamma function) used here will be its **full analytic continuation** to $\mathbb{C} \backslash \mathbb{Z}_{\leqslant 0}$ (resp. $\mathbb{C}$).

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- The gamma function (resp. reciprocal gamma function) used here will be its **full analytic continuation** to $\mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$ (resp. $\mathbb{C}$).

Idea that yields a form of umbral calculus:

$$\Gamma(z) = \int_0^\infty dt\, \frac{1}{t} e^{-t} \boxed{t^z}\,, \qquad \frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_\gamma dt\, e^t \boxed{t^{-z}}$$

Alphabets and generalized polynomials

Definition

Let $\mathscr{A}$ be an **alphabet** (i.e., a set of symbols or of formal variables), and let $\mathbb{C}^{(\mathscr{A})}$ denote the set of all functions $\alpha : \mathscr{A} \to \mathbb{C}$ with **finite support**,

$$\mathbb{C}^{(\mathscr{A})} := \{\alpha : \mathscr{A} \to \mathbb{C} \,|\, |supp(\alpha)| < \infty\} \tag{23}$$

Then we denote by $G_{\mathbb{C}}(\mathscr{X}) \equiv (\mathbb{C}^{(\mathscr{A})}, \cdot)$ the group of **generalized monomials**, with **multiplication**

$$\mathscr{A}^{\alpha} \cdot \mathscr{A}^{\beta} = \mathscr{A}^{\alpha+\beta}\,, \tag{24}$$

where we employ the **multi-index notation**

$$\{\mathscr{A}^{\alpha}\}_{\alpha \in \mathbb{C}^{(\mathscr{A})}}\,. \tag{25}$$

Moreover, we denote by $\mathbb{C}[G_{\mathbb{C}}(\mathscr{A})]$ the **$\mathbb{C}$-algebra** of the group $G_{\mathbb{C}}(\mathscr{A})$.

Thus for a given function $\alpha : \mathscr{A} \to \mathbb{C}$ with support

$$supp(\alpha) = \{a_1, \ldots, a_n\} \subset \mathscr{A}$$

the corresponding element $\mathscr{A}^{\alpha} \in G_{\mathbb{C}}(\mathscr{A})$ reads more explicitly

$$\mathscr{A}^{\alpha} = a_1^{\alpha(a_1)} \ldots a_n^{\alpha(a_n)}\,.$$

Definition

Let $\mathscr{A} = \{\lambda\} \uplus \mathscr{U} \uplus \mathscr{V} \uplus \mathscr{X}$ be an **alphabet of formal variables**, where

- $\uplus$ denotes the operation of **disjoint union**, and
- where $\{\lambda\}$, $\mathscr{U}$, $\mathscr{V}$ and $\mathscr{X}$ are **four (disjoint) alphabets of auxiliary formal variables**.

Let furthermore $\mathscr{A}_\bullet = \mathscr{A}\backslash\{\lambda\}$.
We define a **formal integration operator** $\hat{\mathbb{I}}$ via specifying first its domain $dom(\hat{\mathbb{I}})$ as

$$dom(\hat{\mathbb{I}}) := \{S \in \mathbb{C}[G_{\mathbb{C}}(\mathscr{A}_\bullet)][[\lambda]] \mid \text{for all } \mathscr{A}^{\alpha} \in supp(S) : range(\alpha|_{\mathscr{U}}) \subset \mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}\} \tag{26}$$

whence elements of $dom(\hat{\mathbb{I}})$ are formal power series in λ with **coefficients that are generalized polynomials** over the alphabet $\mathscr{A}_\bullet$ (where the extension to formal power series requires a suitable notion of summability, see the paper).

Definition

Let $\mathscr{A} = \{\lambda\} \uplus \mathscr{U} \uplus \mathscr{V} \uplus \mathscr{X}$ be an **alphabet of formal variables**, and let $\mathscr{A}_\bullet = \mathscr{A} \setminus \{\lambda\}$.
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$$dom(\hat{\mathbb{I}}) := \{S \in \mathbb{C}[G_{\mathbb{C}}(\mathscr{A}_\bullet)][[\lambda]] \mid \text{for all } \mathscr{A}^\alpha \in supp(S) : range(\alpha|_{\mathscr{U}}) \subset \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}\}\,. \tag{26}$$

Then for some monomial $\mathscr{A}^\alpha \in dom(\hat{\mathbb{I}})$, the action of $\hat{\mathbb{I}}$ on $\mathscr{A}^\alpha$ is defined as

$$\begin{aligned}
\hat{\mathbb{I}}(\mathscr{A}^\alpha) &:= \left(\prod_{u_i \in U(\alpha)} \int_0^\infty du_i\, \frac{e^{-u_i}}{u_i}\right)\left[\left(\prod_{v_j \in V(\alpha)} \frac{1}{2\pi i}\int_\gamma dv_j\, e^{v_j}\right)\left[\mathscr{A}^{\tilde{\alpha}}\right]\right] \\
U(\alpha) &:= supp(\alpha) \cap \mathscr{U}\,,\; V(\alpha) := supp(\alpha) \cap \mathscr{V} \\
\tilde{\alpha}(a) &:= \begin{cases} \alpha(a) & \text{if } a \in \mathscr{A} \setminus \mathscr{V} \\ -\alpha(a) & \text{if } a \in \mathscr{V} \end{cases}
\end{aligned} \tag{27}$$

We extend $\hat{\mathbb{I}}$ by linearity to finite sums. For **infinite sums**, this requires an appropriate notion of convergence. A series $\sum_{i \in I} c_i \mathscr{A}^{\alpha_i}$ will be in $dom(\hat{\mathbb{I}})$ (the domain of $\hat{\mathbb{I}}$) if the family $\left(c_i \hat{\mathbb{I}}(\mathscr{A}^{\alpha_i})\right)_{i \in I}$ is **summable** in the target (in the sense of discrete summability or compact convergence for entire functions).

Illustrating a key property of multi-variate umbral calculus

Lemma

We suppose that

(i) the alphabet $\mathscr{A}$ is partitioned into **two disjoint subalphabets**

$$\mathscr{A} = \mathscr{A}_1 \uplus \mathscr{A}_2$$

(ii) we are given two series

$$S_1 = \sum_{i \in I_1} c_{\alpha_i} \mathscr{A}_1^{\alpha_i} \text{ and } S_2 = \sum_{j \in I_2} c_{\alpha_j} \mathscr{A}_2^{\alpha_j}$$

such that $S_1, S_2 \in dom(\hat{\mathbb{I}})$.

Then $\hat{\mathbb{I}}(S_1 S_2)$ exists and

$$\hat{\mathbb{I}}(S_1 S_2) = \hat{\mathbb{I}}(S_1)\hat{\mathbb{I}}(S_2)\,.$$

Collection of first examples

- For $\alpha \in \mathbb{C} \backslash \mathbb{Z}_{\leqslant 0}$ and $\beta \in \mathbb{C}$,

$$\hat{\mathbb{I}}\left(u^{\alpha}\right) = \Gamma(\alpha), \quad \hat{\mathbb{I}}\left(v^{\beta}\right) = \frac{1}{\Gamma(\beta)}. \tag{28}$$

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- For $\alpha, \beta \in \mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$ and $n \in \mathbb{Z}_{\geqslant 0}$,

$$\hat{\mathbb{I}}\,(u^{\alpha+n}v^{\alpha}) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = (\alpha)_n\,, \quad \hat{\mathbb{I}}\left(u^{\beta}v^{\beta+n}\right) = \frac{\Gamma(\beta)}{\Gamma(\beta+n)} = \frac{1}{(\beta)_n}\,. \tag{29}$$

  Here, we employ the notation $(x)_y := \Gamma(x+y)/\Gamma(x)$ for the **Pochhammer symbol**, as customary e.g., in the literature on hypergeometric functions.

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  Here, we employ the notation $(x)_y := \Gamma(x+y)/\Gamma(x)$ for the **Pochhammer symbol**, as customary e.g., in the literature on hypergeometric functions.

- Alternative formula for the **generalized hypergeometric functions**, with parameters $\alpha_i, \beta_j \in \mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$ and with **formal** variable z:

$$ {}_pF_q\left[\begin{matrix}\underline{\alpha} \\ \underline{\beta}\end{matrix}; z\right] \equiv {}_pF_q\left[\begin{matrix}(\alpha_i)_{1\leqslant i\leqslant p} \\ (\beta_j)_{1\leqslant j\leqslant q}\end{matrix}; z\right] := \sum_{n\geqslant 0} \frac{z^n}{n!}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n} \tag{30}$$

$$= \hat{\mathbb{I}}\left(\left(\prod_{i=1}^{p}(u_i v_i)^{\alpha_i}\right)\left(\prod_{j=1}^{q}(u_{j+p}v_{j+p})^{\beta_j}\right)e^{zu_1\cdots u_p v_{p+1}\cdots v_{p+q}}\right) \tag{31}$$

Collection of first examples

- For $\alpha \in \mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$ and $\beta \in \mathbb{C}$,

$$\hat{\mathbb{I}}\left(u^{\alpha}\right) = \Gamma(\alpha)\,, \quad \hat{\mathbb{I}}\left(v^{\beta}\right) = \frac{1}{\Gamma(\beta)}\,. \tag{32}$$

- **eigenfunction** of ∂_x vs. eigenfunction of ${}_L\partial_x := \partial_x x \partial_x$ (**Laguerre derivative**):

$$\partial_x e^{\lambda x} = \lambda e^{\lambda x}\,, \qquad \partial_x x \partial_x \hat{\mathbb{I}}\left(v e^{v\lambda x}\right) = \partial_x x \partial_x \sum_{n\geqslant 0} \frac{x^n}{(n!)^2} = \lambda \hat{\mathbb{I}}\left(v e^{v\lambda x}\right) \tag{33}$$

$\Rightarrow$ this is a first example of an **umbral image** type result!

Discoveries and results on Sobolev-Jacobi polynomials and their lacunary exponential generating functions

Table of Sobolev-Jacobi vs. two-variable Hermite polynomials for $n = 0, \cdots, 10$

n	$\tilde{P}_n^{(-1,-1)}(x)$	$H_n(x,y)$
0	1	1
1	x	x
2	$x^2 - 1$	$x^2 + 2y$
3	$x^3 - x$	$x^3 + 6xy$
4	$x^4 - \frac{6x^2}{5} + \frac{1}{5}$	$x^4 + 12x^2y + 12y^2$
5	$x^5 - \frac{10x^3}{7} + \frac{3x}{7}$	$x^5 + 20x^3y + 60xy^2$
6	$x^6 - \frac{5x^4}{3} + \frac{5x^2}{7} - \frac{1}{21}$	$x^6 + 30x^4y + 180x^2y^2 + 120y^3$
7	$x^7 - \frac{21x^5}{11} + \frac{35x^3}{33} - \frac{5x}{33}$	$x^7 + 42x^5y + 420x^3y^2 + 840xy^3$
8	$x^8 - \frac{28x^6}{13} + \frac{210x^4}{143} - \frac{140x^2}{429} + \frac{5}{429}$	$x^8 + 56x^6y + 840x^4y^2 + 3360x^2y^3 + 1680y^4$
9	$x^9 - \frac{12x^7}{5} + \frac{126x^5}{65} - \frac{84x^3}{143} + \frac{7x}{143}$	$x^9 + 72x^7y + 1512x^5y^2 + 10080x^3y^3 + 15120xy^4$
10	$x^{10} - \frac{45x^8}{17} + \frac{42x^6}{17} - \frac{210x^4}{221} + \frac{315x^2}{2431} - \frac{7}{2431}$	$x^{10} + 90x^8y + 2520x^6y^2 + 25200x^4y^3 + 75600x^2y^4 + 30240y^5$

Sobolev-Jacobi polynomials as umbral images of bi-variate Hermite polynomials [5]

Consider the polynomials $P_n(x,y)$ and $H_n(x,z)$ and their exponential generating functions:

$$P_n(x,y) := e^{-\frac{1}{2}\frac{1}{\hat{D}_x+\hat{D}_y-1}\partial_x^2}(xy)^n\,, \qquad \mathscr{G}(\lambda;x,y) := \sum_{n\geqslant 0}\frac{\lambda^n}{n!}P_n(x,y)$$

$$H_n(x,z) := e^{z\partial_x^2}x^n\,, \qquad \mathscr{H}(\lambda;x,z) := \sum_{n\geqslant 0}\frac{\lambda^n}{n!}H_n(x,z)\,.$$

Theorem 1 of [5]

The exponential generating function $\mathscr{G}(\lambda;x,y)$ is an **umbral image** of the generating function $\mathscr{H}(\lambda;x,z)$,

$$\mathscr{G}(\lambda;x,y) = \hat{\mathbb{I}}\left(\frac{1}{\sqrt{uv}}e^{(\lambda uvy)x+(\lambda uvy)^2\left(-\frac{1}{4u}\right)}\right) = \hat{\mathbb{I}}\left(\frac{1}{\sqrt{uv}}\mathscr{H}\left(\lambda uvy;x,-\tfrac{1}{4u}\right)\right)\,. \qquad (34)$$

Thus the **monic SJ polynomials** $P_n(x) \equiv \tilde{P}_n^{(-1,-1)}(x) := P_n(x,1)$ may alternatively be expressed as

$$P_n(x) \equiv \tilde{P}_n^{(-1,-1)}(x) = \hat{\mathbb{I}}\left((uv)^{n-\frac{1}{2}}H_n(x,-\tfrac{1}{4u})\right) = \hat{\mathbb{I}}\left(\frac{1}{\sqrt{uv}}e^{-\frac{1}{4u}\partial_x^2}(xuv)^n\right)\,. \qquad (35)$$

[5] Nicolas Behr, Giuseppe Dattoli, Gérard H. E. Duchamp, Silvia Licciardi, and Karol A. Penson. “Operational Methods in the Study of Sobolev-Jacobi Polynomials”. In: *Mathematics* 7.2 (2019)

All-order lacunary exponential generating functions for the SJ polynomials

Consider the polynomials $P_n(x,y)$ and $H_n(x,z)$ and their **lacunary EGFs** ($K \geqslant 1, L \geqslant 0$):

$$\mathscr{G}_{K,L}(\lambda;x,y) := \sum_{n\geqslant 0} \frac{\lambda^n}{n!} P_{n\cdot K+L}(x,y)\,, \qquad \mathscr{H}_{K,L}(\lambda;x,z) := \sum_{n\geqslant 0} \frac{\lambda^n}{n!} H_{n\cdot K+L}(x,z)\,.$$

Theorem 2 of [5]

$$\mathscr{G}_{K=2T,0}(\lambda;x) = \sum_{\beta=0}^{T-1} \sum_{s\geqslant 0} \frac{\lambda^s}{s!} x^{K\cdot s-2\beta} \left(-\tfrac{1}{4}\right)^{\beta} \tilde{h}_{K\cdot s,\beta} \frac{\Gamma(K\cdot s-\beta-\frac{1}{2})}{\Gamma(K\cdot s-\frac{1}{2})} \cdot$$

$$\cdot\; {}_{(3T-1)}F_{(3T-1)}\left[\begin{matrix} \left(\left(s+\frac{j+1}{K}\right)_{0\leqslant j\leqslant K-2},\ \left(2s+\frac{2m-\beta-1}{K}\right)_{0\leqslant m\leqslant T-1}\right) \\ \left(\left(\frac{\beta+\ell+1}{T}\right)_{\substack{0\leqslant \ell\leqslant T-1,\\ \ell\neq T-1-\beta}},\ \left(s+\frac{2t-1}{2K}\right)_{0\leqslant t\leqslant K-1}\right)\end{matrix} ;\lambda\left(-\tfrac{1}{4}\right)^{T}\right]$$

Here, $\tilde{h}_{n,k}$ denote the so-called **matching coefficients** (of directed Hermite-configurations),

$$\tilde{h}_{n,k} := \begin{cases} \frac{n!}{(n-2k)!k!} & \text{if } 0 \leqslant 2k \leqslant n \\ 0 & \text{otherwise.}\end{cases}$$

[5] Nicolas Behr, Giuseppe Dattoli, Gérard H. E. Duchamp, Silvia Licciardi, and Karol A. Penson. "Operational Methods in the Study of Sobolev-Jacobi Polynomials". In: *Mathematics* 7.2 (2019)

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$$\cdot\ {}_{(3K-2)}F_{(3K-1)}\left[\begin{matrix} \left(\left(\frac{s}{2}+\frac{j+1}{2K}\right)_{\substack{0\leqslant j\leqslant 2K-2\\ j\neq K-1}},\ \left(s+\frac{2m-2\beta-1}{2K}\right)_{0\leqslant m\leqslant K-1}\right) \\ \left(\left(\frac{\beta+\ell+1}{K}\right)_{\substack{0\leqslant \ell\leqslant K-1\\ \ell\neq K-1-\beta}},\ \left(\frac{s}{2}+\frac{2t-1}{4K}\right)_{0\leqslant t\leqslant 2K-1}\right)\end{matrix};-\frac{\lambda^2}{4^{K+1}}\right]$$

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Theorem 2 of [5]

For integer parameters $K \in \mathbb{Z}_{\geqslant 1}$, denote by

$$\mathscr{G}_K(\mu;\lambda;x) := \sum_{L\geqslant 0} \frac{\mu^L}{L!} \mathscr{G}_{K,L}(\lambda;x) \tag{36}$$

the exponential generating function of lacunary shifts of the lacunary generating functions $\mathscr{G}_{K,0}(\lambda;x)$. Then $\mathscr{G}_K(\mu;\lambda;x)$ is given by

$$\mathscr{G}_K(\mu;\lambda;x) = \hat{\mathbb{I}}\left(\frac{1}{\sqrt{uv}}\, e^{\mu uvx - \frac{\mu^2 uv^2}{4}}\; \mathscr{H}_{K,0}\left(\lambda\,(uv)^K; x - \frac{\mu v}{2}, -\frac{1}{4u}\right)\right)\,. \tag{37}$$

[5] Nicolas Behr, Giuseppe Dattoli, Gérard H. E. Duchamp, Silvia Licciardi, and Karol A. Penson. "Operational Methods in the Study of Sobolev-Jacobi Polynomials". In: *Mathematics* 7.2 (2019)

Examples for SJ-polynomial lacunary exponential generating functions

$$\mathscr{G}_{K,L}(\lambda;x) := \sum_{n\geqslant 0} \frac{\lambda^n}{n!} P_{n\cdot K+L}(x)$$

$$\mathscr{G}_{1,0}(\lambda;x) = \sum_{s\geqslant 0} \frac{\lambda^s}{s!} x^s \, {}_1F_2\left[\begin{matrix} s-\frac{1}{2} \\ \frac{s}{2}-\frac{1}{4},\ \frac{s}{2}+\frac{1}{4} \end{matrix};-\frac{\lambda^2}{16}\right]$$

$$\mathscr{G}_{2,0}(\lambda;x) = \sum_{s\geqslant 0} \frac{\lambda^s}{s!} x^{2s} \, {}_2F_2\left[\begin{matrix} s+\frac{1}{2},\ 2s-\frac{1}{2} \\ s-\frac{1}{4},\ s+\frac{1}{4} \end{matrix};-\frac{\lambda}{4}\right]$$

$$\mathscr{G}_{2,1}(\lambda;x) = \sum_{s\geqslant 0} \frac{\lambda^s}{s!} x^{2s+1} \, {}_2F_2\left[\begin{matrix} s+\frac{3}{2},\ 2s+\frac{1}{2} \\ s+\frac{1}{4},\ s+\frac{3}{4} \end{matrix};-\frac{\lambda}{4}\right]$$

Examples for SJ-polynomial lacunary exponential generating functions

$$\mathscr{G}_{3,0}(\lambda;x) = \sum_{s\geqslant 0} \frac{\lambda^s}{s!}\, x^{3s}\; {}_7F_8\left[\begin{matrix} \frac{s}{2}+\frac{1}{6},\ \frac{s}{2}+\frac{1}{3},\ \frac{s}{2}+\frac{2}{3},\ \frac{s}{2}+\frac{5}{6},\ s-\frac{1}{6},\ s+\frac{1}{6},\ s+\frac{1}{2} \\ \frac{1}{3},\ \frac{2}{3},\ \frac{s}{2}-\frac{1}{12},\ \frac{s}{2}+\frac{1}{12},\ \frac{s}{2}+\frac{1}{4},\ \frac{s}{2}+\frac{5}{12},\ \frac{s}{2}+\frac{7}{12},\ \frac{s}{2}+\frac{3}{4} \end{matrix};-\frac{\lambda^2}{256}\right]$$

$$-\sum_{s\geqslant 0} \frac{\lambda^{s+1}}{(s+1)!}\, x^{3(s+1)-2} \left(\frac{\Gamma\left(3(s+1)-\frac{3}{2}\right)}{4\Gamma\left(3(s+1)-\frac{1}{2}\right)}\right)\left(\frac{(3(s+1))!}{(3(s+1)-2)!}\right)\cdot$$

$$\cdot\ {}_7F_8\left[\begin{matrix} \frac{s+1}{2}+\frac{1}{6},\ \frac{s+1}{2}+\frac{1}{3},\ \frac{s+1}{2}+\frac{2}{3},\ \frac{s+1}{2}+\frac{5}{6},\ (s+1)-\frac{1}{2},\ (s+1)-\frac{1}{6},\ (s+1)+\frac{1}{6} \\ \frac{2}{3},\ \frac{4}{3},\ \frac{s+1}{2}-\frac{1}{12},\ \frac{s+1}{2}+\frac{1}{12},\ \frac{s+1}{2}+\frac{1}{4},\ \frac{s+1}{2}+\frac{5}{12},\ \frac{s+1}{2}+\frac{7}{12},\ \frac{s+1}{2}+\frac{3}{4} \end{matrix};-\frac{\lambda^2}{256}\right]$$

$$+\sum_{s\geqslant 0} \frac{\lambda^{s+2}}{(s+2)!}\, x^{3(s+2)-4} \left(\frac{\Gamma\left(3(s+2)-\frac{5}{2}\right)}{16\Gamma\left(3(s+2)-\frac{1}{2}\right)}\right)\left(\frac{(3(s+2))!}{2!(3(s+2)-4)!}\right)\cdot$$

$$\cdot\ {}_7F_8\left[\begin{matrix} \frac{s+2}{2}+\frac{1}{6},\ \frac{s+2}{2}+\frac{1}{3},\ \frac{s+2}{2}+\frac{2}{3},\ \frac{s+2}{2}+\frac{5}{6},\ (s+2)-\frac{5}{6},\ (s+2)-\frac{1}{2},\ (s+2)-\frac{1}{6} \\ \frac{4}{3},\ \frac{5}{3},\ \frac{s+2}{2}-\frac{1}{12},\ \frac{s+2}{2}+\frac{1}{12},\ \frac{s+2}{2}+\frac{1}{4},\ \frac{s+2}{2}+\frac{5}{12},\ \frac{s+2}{2}+\frac{7}{12},\ \frac{s+2}{2}+\frac{3}{4} \end{matrix};-\frac{\lambda^2}{256}\right]$$

Examples for SJ-polynomial lacunary exponential generating functions

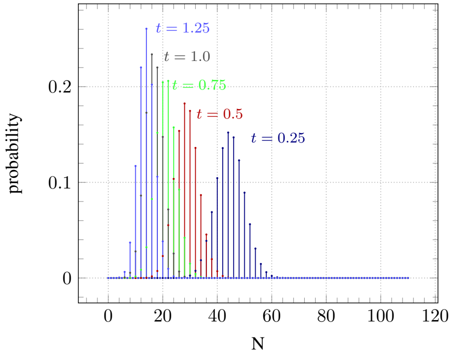
$$\mathscr{G}_{4,0}(\lambda;x) = \sum_{s\geqslant 0} \frac{\lambda^s}{s!}\, x^{4s}\, {}_5F_5\left[\begin{matrix} s+\frac{1}{4},\ s+\frac{1}{2},\ s+\frac{3}{4},\ 2s-\frac{1}{4},\ 2s+\frac{1}{4} \\ \frac{1}{2},\ s-\frac{1}{8},\ s+\frac{1}{8},\ s+\frac{3}{8},\ s+\frac{5}{8} \end{matrix};\frac{\lambda}{16}\right]$$

$$-\sum_{s\geqslant 0} \frac{\lambda^{s+1}}{(s+1)!}\, x^{4(s+1)-2} \left(\frac{\Gamma\big(4(s+1)-\frac{3}{2}\big)}{4\Gamma\big(4(s+1)-\frac{1}{2}\big)}\right) \left(\frac{(4(s+1))!}{(4(s+1)-2)!}\right) \cdot$$

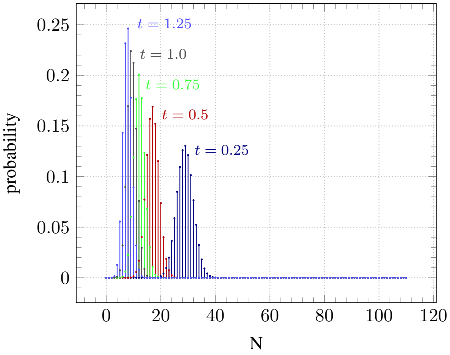
$$\cdot\ {}_5F_5\left[\begin{matrix} (s+1)+\frac{1}{4},\ (s+1)+\frac{1}{2},\ (s+1)+\frac{3}{4},\ 2(s+1)-\frac{3}{4},\ 2(s+1)-\frac{1}{4} \\ \frac{3}{2},\ (s+1)-\frac{1}{8},\ (s+1)+\frac{1}{8},\ (s+1)+\frac{3}{8},\ (s+1)+\frac{5}{8} \end{matrix};\frac{\lambda}{16}\right]$$

Elementary binary reactions – plots [5]

e) pair annihilation reaction $2A \xrightarrow{\kappa=\frac{1}{40}} 0A$



f) catalytic decay reaction $2A \xrightarrow{\lambda=\frac{1}{10}} 1A$



[5] Nicolas Behr, Gérard H.E. Duchamp, and Karol A. Penson. “Combinatorics of Chemical Reaction Systems”. In: *arXiv:1712.06575* (2017)

OPSFA 15 “mini-challenge”

How does one compute in closed form the probability generating function $P(t;x)$ with **evolution equation**

$$\partial_t P(t;x) = \left[\alpha(\hat{x}^2 - 1) + \beta(1 - \hat{x}^2)\partial_x^2\right] P(t;x)$$

$$P(0;x) = x^n$$

describing the system with reactions $\varnothing \xrightarrow{\alpha} 2X$ (**pair creation**) and $2X \xrightarrow{\beta} \varnothing$ (**pair annihilation**)?

Here, $\alpha, \beta \in \mathbb{R}_{>0}$ are non-zero real parameters (**reaction rates**), and $\hat{x}[x^n] := x^{n+1}$.

Thank you!

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