Explicit formulae for all higher order exponential lacunary generating functions of Hermite polynomials

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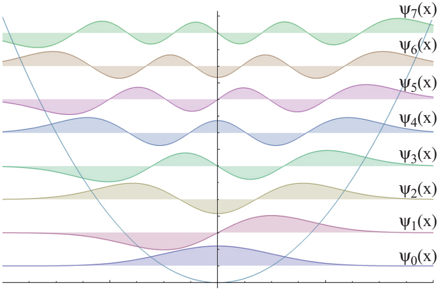
18th Workshop: Noncommutative Probability, Operator Algebras, Random Matrices and Related Topics, with Applications

Stefan Banach Conference Center of the Polish Academy of Sciences, Będlewo,
July 16th 2018

Hermite polynomials in physics: the quantum harmonic oscillator



Energy levels, harmonic oscillator potential and the first eight eigenfunctions vs. the position coordinate (*source: Fig. 3.4-1 of* [1])

- two-variable **Hermite** (or so-called Hermite-Kampé de Fériet) **polynomials** [2] [3]:

$$H_n(x,y) := e^{y\partial_x^2} x^n \quad (n \in \mathbb{Z}_{\geqslant 0})$$

- **quantum harmonic oscillator:** Schrödinger equation

$$i\hbar\partial_t \Psi(x,t) = -\frac{\hbar^2}{2m}\partial_x^2 \Psi(x,t) + \frac{1}{2}kx^2\Psi(x,t)$$

$\Rightarrow$ stationary solutions are of the form

$$\Psi(x,t) = e^{-i\frac{E_n t}{\hbar}}\Phi_n(x)\,, \quad \Phi_n(x) = \frac{H_n(x)e^{-x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}}$$

with $H_n(x) := H_n(2x,-1)$

[1] D Babusci, G Dattoli, and M Del Franco. "Lectures on mathematical methods for physics". In: *Thecnical Report* 58 (2010)

[2] Paul Appell and Joseph Kampé de Fériet. *Fonctions hypergéométriques et hypersphériques : polynomes d'Hermite*. French. Bibliography: p. 419-427. Paris : Gauthier-Villars, 1926

[3] Giuseppe Dattoli et al. "Evolution operator equations: Integration with algebraic and finitedifference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory". In: *La Rivista del Nuovo Cimento (1978-1999)* 20.2 (1997), p. 3

Hermite polynomials in quantum probability [4]

- recalling that $H_n(x) := H_n(2x,-1) = e^{-\partial_x^2}(2x)^n$, consider the following variant:

$$\tilde{H}_n(x) := \left(\frac{1}{\sqrt{2}}\right)^n H_n\left(\frac{x}{\sqrt{2}}\right) = \left(\frac{1}{\sqrt{2}}\right)^n e^{-\partial_x^2}\left(\frac{2x}{\sqrt{2}}\right) = e^{-\partial_x^2}x^n$$

- the **orthogonality relation** for the polynomials $\tilde{H}_n(x)$ reads

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \tilde{H}_m(x)\tilde{H}_n(x)e^{-x^2/2}dx = n!\delta_{m,n}\,,$$

  whence these polynomials are orthogonal w.r.t. the **standard normal distribution**

- let Γ_0 be the dense subspace of a Hilbert space Γ spanned by an orthonormal basis $\{\Phi_0, \Phi_1, \ldots\}$ of Γ, let $\langle , \rangle$ denote the inner product on Γ, and define the linear operators $B^+, B^0, B^- \in \mathcal{L}(\Gamma_0)$:

$$B^+\Phi_n := \sqrt{\omega_{n+1}}\Phi_{n+1}\ (n \geqslant 0)\,,\ B^0\Phi_n := \alpha_{n+1}\Phi_n\ (n \geqslant 0)\,,\ B^-\Phi_n := \begin{cases} 0 & \text{if } n = 0 \\ \sqrt{\omega_n}\Phi_{n-1} & \text{if } n \geqslant 1 \end{cases}$$

  then for **Jacobi parameters** $\{\omega_n = n\}_{n\in\mathbb{Z}_{\geqslant 0}}$ and $\{\alpha_n = n\}_{n\in\mathbb{Z}_{\geqslant 0}}$,

$$\langle \Phi_0, (B^+ + B^-)^m\Phi_0\rangle = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} x^m e^{-x^2/2}dx$$

[4] Nobuaki Obata. *Spectral analysis of growing graphs: a quantum probability point of view.* Vol. 20. Springer, 2017

“Lacunary generating-functionology”: general results

Exponential generating functions

- let $\{p_n(x,y)\}_{n\in\mathbb{Z}_{\geqslant 0}}$ be a **sequence of polynomials** with $\deg_x(p_n(x,y)) = n$
- typically, we will consider the **variable** y as a **formal parameter**, and focus on expansions in terms of the variable x
- given a particular set of polynomials, one may compute the **exponential generating function** $\mathcal{G}(\lambda;x,y)$ for this set as

$$\begin{align}
\mathcal{G}(\lambda;x,y) &:= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, p_n(x,y) \tag{1a}\\
&= \sum_{r=0}^{\infty} x^r \sum_{m=0}^{\infty} \frac{\lambda^{r+m}}{(r+m)!}\, g_{r,m}(y)\,. \tag{1b}
\end{align}$$

- while the form as presented in (1a) merely amounts to re-encoding of the information available via the explicit definition of the polynomials $p_n(x,y)$, the form (1b) in fact **necessitates a calculation**: equation (1a) must be **expanded into powers of** x and then further **in powers of the formal variable** λ, which (in the specific **pairing** of powers λ^{r+m} with $1/(r+m)!$) defines a set of expansion coefficients $g_{r,m}(y)$

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$$
\begin{aligned}
\mathcal{G}(\lambda; x, y) &:= \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, p_n(x,y) && \text{(1a)}\\
&= \sum_{r=0}^{\infty} x^r \sum_{m=0}^{\infty} \frac{\lambda^{r+m}}{(r+m)!}\, g_{r,m}(y)\,. && \text{(1b)}
\end{aligned}
$$

- **example:** for the **Hermite polynomials** $H_n(x,y)$, it is well-known that

$$
\begin{aligned}
\mathcal{H}(\lambda; x, y) &:= \sum_{n\geqslant 0} \frac{\lambda^n}{n!} H_n(x,y) = e^{\lambda x + \lambda^2 y} = \sum\\
&= \sum_{r=0}^{\infty} x^r \sum_{m=0}^{\infty} \frac{\lambda^{r+2m}}{r!m!} y^m = \sum_{r=0}^{\infty} x^r \sum_{m=0}^{\infty} \frac{\lambda^{r+2m}}{(r+2m)!} \left(\frac{(r+2m)!\, y^m}{r!m!}\right)
\end{aligned}
$$

$$
\Rightarrow \quad g_{r,2s} = \frac{(r+2s)!y^s}{r!s!} \text{ and } g_{r,2s+1} = 0 \quad (s \geqslant 0)
$$

An important technicality: the notion of summability

- polynomials or series of any sort are functions $M \to k$ where M is a **set of monomials** (a **monoid**, since the product of two monomials is a monomial, and the void monomial is the neutral element), and k its **set of coefficients** ($\mathbb{R}, \mathbb{C}$, any ring as a ring of polynomials or series)
- in order to manipulate safely **infinite** sums, such as in the transition from (1a) to the form (1b), these spaces are endowed with the folowing notion:

Summability

A family of series $(S_i)_{i\in I}$ is said to be **summable** [5] if it is **locally finite** [6], i.e. if

$$(\forall m \in M)\Big(\ |\{i \in I | \langle S_i \mid m\rangle \neq 0\}| < \infty\ \Big)\,,$$

where $\langle S_i \mid m\rangle$ stands for **"the coefficient of the monomial m in S_i"**.
In this case ($(S_i)_{i\in I}$ is summable), we say that $S = \sum_{i\in I} S_i$ where, for all $m \in M$

$$\langle S \mid m\rangle = \sum_{i\in I} \langle S_i \mid m\rangle\,.$$

[5] Christophe Reutenauer. *Free Lie algebras, volume 7 of London Mathematical Society Monographs. New Series*. 1993

[6] Jean Berstel and Christophe Reutenauer. *Rational series and their languages*. Vol. 12. Springer-Verlag, 1988

Lacunary exponential generating functions, shifts and dilatations

- given a set of polynomials $p_n(x,y)$ as before, the so-called **K-tuple L-shifted lacunary generating functions** $\mathcal{G}_{K,L}(\lambda;x,y)$ (for $K \in \mathbb{Z}_{\geqslant 1}$ and $L \in \mathbb{Z}_{\geqslant 0}$) are defined as

$$\mathcal{G}_{K,L}(\lambda;x,y) := \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, p_{K\cdot n+L}(x,y) \tag{2a}$$

$$= \sum_{r=0}^{\infty} x^r \sum_{m=0}^{\infty} \frac{\lambda^{r+m}}{(r+m)!}\, g_{r,\,m}^{(K,L)}(y)\,. \tag{2b}$$

- accordingly, $\mathcal{G}_{1,0}(\lambda;x,y) \equiv \mathcal{G}(\lambda;x,y)$
- **L-fold lacunary shifts:** implemented via $\mathbb{S}_L := \partial_\lambda^L$, since (for all $K \in \mathbb{Z}_{\geqslant 1}$ and $L \in \mathbb{Z}_{\geqslant 0}$)

$$\mathbb{S}_L\left(\mathcal{G}_{K,0}(\lambda;x,y)\right) = \sum_{n=L}^{\infty} \frac{\lambda^{n-L}}{(n-L)!}\, p_{n\cdot K}(x,y) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!}\, p_{n\cdot K+L}(x,y) = \mathcal{G}_{K,L}(\lambda;x,y)$$

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- **K-fold lacunary dilatations** [7]**:** implemented (for $K \in \mathbb{Z}_{\geqslant 1}$) via the (formal) operator $\mathbb{L}_K$, which acts on formal series $F(\lambda)$ via

$$\mathbb{L}_K(F(\lambda)) := F(\lambda)\Big|_{\lambda^n \mapsto\ \delta_{(n \mod K),0}\ \frac{n!}{(n/K)!}\ \lambda^{(n/K)}}$$

$$\Rightarrow \quad \mathbb{L}_K\left(\mathcal{G}_{1,L}(\lambda;x,y)\right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \left(\delta_{(n \mod K),0}\ \frac{n!}{(n/K)!}\ \frac{\lambda^{(n/K)}}{\lambda^n}\right) p_{n+L}(x,y) \tag{3a}$$

$$= \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} p_{r\cdot K+L}(x,y) = \mathcal{G}_{K,L}(\lambda;x,y)\,, \tag{3b}$$

[7] Nicolas Behr, Gérard HE Duchamp, and Karol A Penson. "Explicit formulae for all higher order exponential lacunary generating functions of Hermite polynomials". In: *arXiv preprint arXiv:1806.08417* (2018)

A general formula for K-tuple lacunary generating functions

Lemma [8]

The explicit form for the K-fold lacunary dilatation $\mathbb{L}_K(\mathcal{G}(\lambda; x, y))$ ($K \in \mathbb{Z}_{\geqslant 1}$) of a generating function $\mathcal{G}(\lambda; x, y)$ with expansion coefficients $g_{r,m}(y)$ reads

$$\begin{aligned}
\mathcal{G}_{K,0}(\lambda; x, y) &= \mathbb{L}_K\left(\mathcal{G}(\lambda; x, y)\right) \\
&= \sum_{s=0}^{\infty} x^{s\cdot K} \sum_{q=0}^{\infty} \frac{\lambda^{s+q}}{(s+q)!}\, g_{s\cdot K,\, q\cdot K}(y) \\
&\quad + \sum_{\alpha=1}^{K-1} \sum_{s=0}^{\infty} x^{(s+1)\cdot K-\alpha} \sum_{q=0}^{\infty} \frac{\lambda^{s+q+1}}{(s+q+1)!}\, g_{(s+1)\cdot K-\alpha,\, q\cdot K+\alpha}\,.
\end{aligned} \tag{4}$$

Proof: The argument follows from **splitting the summation** over r in $\mathbb{L}_K(\mathcal{G}(\lambda; x, y))$ (with $\mathcal{G}(\lambda; x, y)$ expanded in the form described in (1b)) **into $\bmod$ K-classes**, which consequently leads to a modification of the summation over the second index m.

[8] Nicolas Behr, Gérard HE Duchamp, and Karol A Penson. “Explicit formulae for all higher order exponential lacunary generating functions of Hermite polynomials”. In: *arXiv preprint arXiv:1806.08417* (2018)

Exponential lacunary generating functions of the Hermite polynomials

Specializing to Hermite-type generating functions

Corollary [9]: refining the Lemma to the case of $g_{r,m}(y) = 0$ for $m = 2s+1$ (i.e. to "part E")

$$\mathbb{L}_{K=2T}\left(\mathcal{G}(\lambda;x,y)\right) = \Bigg[^{\text{part } E} \sum_{s=0}^{\infty} x^{s\cdot K} \sum_{q=0}^{\infty} \frac{\lambda^{s+q}}{(s+q)!}\, g_{s\cdot K,\, q\cdot K}(y) \tag{5a}$$

$$+ \sum_{\beta=1}^{T-1} \sum_{s=0}^{\infty} x^{(s+1)\cdot K-2\beta} \sum_{q=0}^{\infty} \frac{\lambda^{s+q+1}}{(s+q+1)!}\, g_{(s+1)\cdot K-2\beta,\, q\cdot K+2\beta} \Bigg] \tag{5b}$$

$$\mathbb{L}_{K=2T+1}\left(\mathcal{G}(\lambda;x,y)\right) = \Bigg[^{\text{part } E} \sum_{s=0}^{\infty} x^{s\cdot K} \sum_{\ell=0}^{\infty} \frac{\lambda^{s+2\ell}}{(s+2\ell)!}\, g_{s\cdot K,\, 2\ell\cdot K}(y) \tag{5c}$$

$$+ \sum_{\beta=1}^{T} \sum_{s=0}^{\infty} x^{(s+1)\cdot K-2\beta} \sum_{\ell=0}^{\infty} \frac{\lambda^{s+2\ell+1}}{(s+2\ell+1)!}\, g_{(s+1)\cdot K-2\beta,\, 2\ell\cdot K+2\beta} \tag{5d}$$

$$+ \sum_{\beta=1}^{T} \sum_{s=0}^{\infty} x^{(s+1)\cdot K-2\beta+1} \sum_{\ell=0}^{\infty} \frac{\lambda^{s+2\ell+2}}{(s+2\ell+2)!}\, g_{(s+1)\cdot K-2\beta+1,\, (2\ell+1)\cdot K+2\beta-1} \Bigg]. \tag{5e}$$

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[9] Nicolas Behr, Gérard HE Duchamp, and Karol A Penson. "Explicit formulae for all higher order exponential lacunary generating functions of Hermite polynomials". In: *arXiv preprint arXiv:1806.08417* (2018)

Some additional technical tools

- the **Gauss-Legendre multiplication formula** [10] (Eq. 5.5.6) for **gamma functions**,

$$\Gamma(nz) = n^{nz-\frac{1}{2}}(2\pi)^{\frac{(1-n)}{2}}\prod_{j=0}^{n-1}\Gamma\left(z+\frac{j}{n}\right) \quad \text{(for } n\cdot z\notin\mathbb{Z}_{\leqslant 0}\text{)}$$

  will be used in the following **variant** (for $n(s+x), nx \in \mathbb{C}\backslash\mathbb{Z}_{\leqslant 0}$, $n\in\mathbb{Z}_{\geqslant 2}$ and $s\in\mathbb{Z}_{\geqslant 0}$):

$$\Gamma(n(s+x)) = (n^{s\cdot n})\,\Gamma(nx)\prod_{j=0}^{n-1}\left(x+\frac{j}{n}\right)_s\,, \tag{6}$$

  with the **Pochhammer symbol** $(a)_b$ defined according to the convention

$$(a)_b := \frac{\Gamma(a+b)}{\Gamma(a)}\,.$$

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[10] *NIST Digital Library of Mathematical Functions.* http://dlmf.nist.gov/, Release 1.0.18 of 2018-03-27. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

Some additional technical tools

- **notations for hypergeometric functions** [10] [11]:

$$
{}_pF_q\left[\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix}; z\right] := \sum_{s=0}^{\infty} \frac{z^s}{s!} \frac{(a_1)_s \ldots (a_p)_s}{(b_1)_s \ldots (b_q)_s}
$$

- **shorthand notation** for **sequences**: $\left(f_i\right)_{1 \leqslant i \leqslant n}$, such as in

$$
{}_pF_q\left[\begin{matrix} \left(a_i\right)_{1 \leqslant i \leqslant p} \\ \left(b_j\right)_{1 \leqslant i \leqslant q} \end{matrix}; z\right] := {}_pF_q\left[\begin{matrix} a_1, & \ldots, & a_p \\ b_1, & \ldots, & b_q \end{matrix}; z\right]
$$

[10] Anatolii Platonovich Prudnikov, Yurii Aleksandrovich Brychkov, and Oleg Igorevich Marichev. "Integrals and series". In: (1992)

[11] Richard Beals and Roderick Wong. *Special functions and orthogonal polynomials*. Vol. 153. Cambridge University Press, 2016

Some additional technical tools

Proposition [10]

Define the **exponential generating function** of the lacunary shifts $\mathcal{H}_{K,L}(\lambda;x,y)$ of the K-tuple lacunary generating function $\mathcal{H}_{K,0}(\lambda;x,y)$ of the polynomials $H_n(x,y)$ as

$$
\begin{aligned}
\mathcal{R}_K(\mu;\lambda;x,y) &:= \sum_{L=0}^{\infty} \frac{\mu^L}{L!}\mathcal{H}_{K,L}(\lambda;x,y) = \sum_{L=0}^{\infty} \frac{\mu^L}{L!}\mathbb{L}_K\left(\mathcal{H}_{1,L}(\lambda;x,y)\right) \\
&= \sum_{L=0}^{\infty} \frac{\mu^L}{L!}\mathbb{L}_K\left(\left(\tfrac{\partial}{\partial\lambda}\right)^L\left(\mathcal{H}_{1,0}(\lambda;x,y)\right)\right) = \mathbb{L}_K\left(e^{\mu\frac{\partial}{\partial\lambda}}\mathcal{H}_{1,0}(\lambda;x,y)\right).
\end{aligned}
$$

Then by virtue of the **Crofton identity** and of a **semi-linear normal-ordering technique** [11],

$$
\begin{aligned}
&\mathcal{R}_K(\mu;\lambda;x,y) = e^{\mu x+\mu^2 y}\mathcal{H}_{K,0}(\lambda;x+2\mu y,y) = \mathcal{H}_{1,0}(\mu;x,y)\mathcal{H}_{K,0}(\lambda;x+2\mu y,y) \\
\Rightarrow\quad &\mathcal{H}_{K,L}(\lambda;x,y) = \left[\left(\tfrac{\partial}{\partial\mu}\right)^L\mathcal{R}_K(\mu;\lambda;x,y)\right]\Big|_{\mu\mapsto 0}
\end{aligned}
$$

[10] Nicolas Behr, Gérard HE Duchamp, and Karol A Penson. “Explicit formulae for all higher order exponential lacunary generating functions of Hermite polynomials”. In: *arXiv preprint arXiv:1806.08417* (2018)

[11] Giuseppe Dattoli et al. “Evolution operator equations: Integration with algebraic and finitedifference methods. Applications to physical problems in classical and quantum mechanics and quantum field theory”. In: *La Rivista del Nuovo Cimento (1978-1999)* 20.2 (1997), p. 3

All-order lacunary exponential generating functions for the Hermite polynomials

Theorem [12]

The K**-tuple exponential lacunary generating functions** $\mathcal{H}_{K,0}(\lambda; x, y)$ read for $K = 2T$ ($T \in \mathbb{Z}_{\geqslant 1}$)

$$\begin{aligned}
\mathcal{H}_{K=2T,0}(\lambda; x, y) = &\sum_{s=0}^{\infty} \frac{\lambda^s}{s!}\, x^{2Ts}\ {}_{(2T-1)}F_{(T-1)}\left[\begin{matrix}\left(s+\frac{j+1}{2T}\right)_{0\leqslant j\leqslant 2T-2} \\ \left(\frac{\ell+1}{T}\right)_{0\leqslant \ell\leqslant T-2}\end{matrix}; \lambda(4Ty)^T\right] \\
&+ \sum_{\beta=1}^{T-1}\sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\, x^{2T(s+1)-2\beta}\, y^{\beta} \left(\frac{(2T(s+1))!}{(2T(s+1)-2\beta)!\beta!}\right) \cdot \\
&\cdot\ {}_{(2T-1)}F_{(T-1)}\left[\begin{matrix}\left(s+1+\frac{j+1}{2T}\right)_{0\leqslant j\leqslant 2T-2} \\ \left(\frac{\beta+\ell+1}{T}\right)_{\substack{0\leqslant \ell\leqslant T-1 \\ \ell\neq T-1-\beta}}\end{matrix}; \lambda(4Ty)^T\right],
\end{aligned} \tag{6}$$

[12] Nicolas Behr, Gérard HE Duchamp, and Karol A Penson. "Explicit formulae for all higher order exponential lacunary generating functions of Hermite polynomials". In: *arXiv preprint arXiv:1806.08417* (2018)

All-order lacunary exponential generating functions for the Hermite polynomials

Theorem [12]

...and for $K = 2T + 1$ (with $T \in \mathbb{Z}_{\geqslant 1}$)

$$
\begin{aligned}
\mathcal{H}_{K=2T+1,0}(\lambda; x, y) = &\sum_{s=0}^{\infty} \frac{\lambda^s}{s!}\, x^{Ks}\ {}_{(2K-2)}F_{(K-1)}\left[\begin{matrix} \left(\frac{s}{2} + \frac{j+1}{2K}\right)_{\substack{0\leqslant j\leqslant 2K-2\\ j\neq K-1}} \\ \left(\frac{\ell+1}{K}\right)_{0\leqslant \ell\leqslant K-2} \end{matrix};\ \frac{\lambda^2(4Ky)^K}{4}\right] \\
&+ \sum_{\beta=1}^{T}\sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\, x^{K(s+1)-2\beta}\ y^{\beta}\ \left(\frac{(K(s+1))!}{(K(s+1)-2\beta)!\beta!}\right) \cdot \\
&\quad \cdot {}_{(2K-2)}F_{(K-1)}\left[\begin{matrix} \left(\frac{s+1}{2} + \frac{j+1}{2K}\right)_{\substack{0\leqslant j\leqslant 2K-2\\ j\neq K-1}} \\ \left(\frac{\beta+\ell+1}{K}\right)_{\substack{0\leqslant \ell\leqslant K-1\\ \ell\neq K-1-\beta}} \end{matrix};\ \frac{\lambda^2(4Ky)^K}{4}\right] \\
&+ \sum_{\beta=1}^{T}\sum_{s=0}^{\infty} \frac{\lambda^{s+2}}{(s+2)!}\, x^{K(s+2)-2(T+\beta)}\ y^{T+\beta}\ \left(\frac{(K(s+2))!}{(K(s+2)-2(T+\beta))!(T+\beta)!}\right) \cdot \\
&\quad \cdot {}_{(2K-2)}F_{(K-1)}\left[\begin{matrix} \left(\frac{s+2}{2} + \frac{j+1}{2K}\right)_{\substack{0\leqslant j\leqslant 2K-2\\ j\neq K-1}} \\ \left(\frac{T+\beta+\ell+1}{K}\right)_{\substack{0\leqslant \ell\leqslant K-1\\ \ell\neq T-\beta}} \end{matrix};\ \frac{\lambda^2(4Ky)^K}{4}\right] .
\end{aligned}
\tag{6}
$$

[12] Nicolas Behr, Gérard HE Duchamp, and Karol A Penson. “Explicit formulae for all higher order exponential lacunary generating functions of Hermite polynomials”. In: *arXiv preprint arXiv:1806.08417* (2018)

Table 1 Lacunary generating functions $\mathcal{H}_{K,0}(\lambda;x;y)$ for the bi-variate Hermite polynomials $H_n(x,y)$ and $K=1\ldots10$.

$$\mathcal{H}_{1,0}(\lambda;x,y)=e^{\lambda x+\lambda^2 y}$$

$$\mathcal{H}_{2,0}(\lambda;x,y)=\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\,x^{2s}\;{}_1F_0\left[\begin{matrix}s+\frac{1}{2}\\ -\end{matrix};\lambda y\,2^2\right]=\frac{1}{\sqrt{1-4\lambda y}}e^{\frac{x^2\lambda}{\sqrt{1-4\lambda y}}}$$

$$\begin{aligned}\mathcal{H}_{3,0}(\lambda;x,y)=&\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\,x^{3s}\;{}_4F_2\left[\begin{matrix}\frac{s}{2}+\frac{1}{6},\ \frac{s}{2}+\frac{1}{3},\ \frac{s}{2}+\frac{2}{3},\ \frac{s}{2}+\frac{5}{6}\\ \frac{1}{3},\ \frac{2}{3}\end{matrix};\lambda^2y^3\,2^43^3\right]\\ &+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{3s+1}\,y\,\left(\frac{(3(s+1))!}{(3(s+1)-2)!}\right)\;{}_4F_2\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{6},\ \frac{s+1}{2}+\frac{1}{3},\ \frac{s+1}{2}+\frac{2}{3},\ \frac{s+1}{2}+\frac{5}{6}\\ \frac{2}{3},\ \frac{4}{3}\end{matrix};\lambda^2y^3\,2^43^3\right]\\ &+\sum_{s=0}^{\infty}\frac{\lambda^{s+2}}{(s+2)!}\,x^{3s+2}\,y^2\,\left(\frac{(3(s+2))!}{(3(s+2)-4)!2!}\right)\;{}_4F_2\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{6},\ \frac{s+2}{2}+\frac{1}{3},\ \frac{s+2}{2}+\frac{2}{3},\ \frac{s+2}{2}+\frac{5}{6}\\ \frac{4}{3},\ \frac{5}{3}\end{matrix};\lambda^2y^3\,2^43^3\right]\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{4,0}(\lambda;x,y)=&\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\,x^{4s}\;{}_3F_1\left[\begin{matrix}s+\frac{1}{4},\ s+\frac{1}{2},\ s+\frac{3}{4}\\ \frac{1}{2}\end{matrix};\lambda y^2\,2^6\right]\\ &+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{4s+2}\,y\,\left(\frac{(4(s+1))!}{(4(s+1)-2)!}\right)\;{}_3F_1\left[\begin{matrix}(s+1)+\frac{1}{4},\ (s+1)+\frac{1}{2},\ (s+1)+\frac{3}{4}\\ \frac{3}{2}\end{matrix};\lambda y^2\,2^6\right]\end{aligned}$$

$$\begin{aligned}\mathcal{H}_{5,0}(\lambda;x,y)=&\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\,x^{5s}\;{}_8F_4\left[\begin{matrix}\frac{s}{2}+\frac{1}{10},\ \frac{s}{2}+\frac{1}{5},\ \frac{s}{2}+\frac{3}{10},\ \frac{s}{2}+\frac{2}{5},\ \frac{s}{2}+\frac{3}{5},\ \frac{s}{2}+\frac{7}{10},\ \frac{s}{2}+\frac{4}{5},\ \frac{s}{2}+\frac{9}{10}\\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5}\end{matrix};\lambda^2y^5\,2^85^5\right]\\ &+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{5s+3}\,y\,\left(\frac{(5(s+1))!}{(5(s+1)-2)!}\right)\;{}_8F_4\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{10},\ \frac{s+1}{2}+\frac{1}{5},\ \frac{s+1}{2}+\frac{3}{10},\ \frac{s+1}{2}+\frac{2}{5},\ \frac{s+1}{2}+\frac{3}{5},\ \frac{s+1}{2}+\frac{7}{10},\ \frac{s+1}{2}+\frac{4}{5},\ \frac{s+1}{2}+\frac{9}{10}\\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{6}{5}\end{matrix};\lambda^2y^5\,2^85^5\right]\\ &+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{5s+1}\,y^2\,\left(\frac{(5(s+1))!}{(5(s+1)-4)!2!}\right)\;{}_8F_4\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{10},\ \frac{s+1}{2}+\frac{1}{5},\ \frac{s+1}{2}+\frac{3}{10},\ \frac{s+1}{2}+\frac{2}{5},\ \frac{s+1}{2}+\frac{3}{5},\ \frac{s+1}{2}+\frac{7}{10},\ \frac{s+1}{2}+\frac{4}{5},\ \frac{s+1}{2}+\frac{9}{10}\\ \frac{3}{5},\ \frac{4}{5},\ \frac{6}{5},\ \frac{7}{5}\end{matrix};\lambda^2y^5\,2^85^5\right]\\ &+\sum_{s=0}^{\infty}\frac{\lambda^{s+2}}{(s+2)!}\,x^{5s+4}\,y^3\,\left(\frac{(5(s+2))!}{(5(s+2)-6)!3!}\right)\;{}_8F_4\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{10},\ \frac{s+2}{2}+\frac{1}{5},\ \frac{s+2}{2}+\frac{3}{10},\ \frac{s+2}{2}+\frac{2}{5},\ \frac{s+2}{2}+\frac{3}{5},\ \frac{s+2}{2}+\frac{7}{10},\ \frac{s+2}{2}+\frac{4}{5},\ \frac{s+2}{2}+\frac{9}{10}\\ \frac{4}{5},\ \frac{6}{5},\ \frac{7}{5},\ \frac{8}{5}\end{matrix};\lambda^2y^5\,2^85^5\right]\\ &+\sum_{s=0}^{\infty}\frac{\lambda^{s+2}}{(s+2)!}\,x^{5s+2}\,y^4\,\left(\frac{(5(s+2))!}{(5(s+2)-8)!4!}\right)\;{}_8F_4\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{10},\ \frac{s+2}{2}+\frac{1}{5},\ \frac{s+2}{2}+\frac{3}{10},\ \frac{s+2}{2}+\frac{2}{5},\ \frac{s+2}{2}+\frac{3}{5},\ \frac{s+2}{2}+\frac{7}{10},\ \frac{s+2}{2}+\frac{4}{5},\ \frac{s+2}{2}+\frac{9}{10}\\ \frac{6}{5},\ \frac{7}{5},\ \frac{8}{5},\ \frac{9}{5}\end{matrix};\lambda^2y^5\,2^85^5\right]\end{aligned}$$

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$$\mathcal{H}_{6,0}(\lambda;x,y)=\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\,x^{6s}\;{}_5F_2\left[{s+\frac{1}{6},\ s+\frac{1}{3},\ s+\frac{1}{2},\ s+\frac{2}{3},\ s+\frac{5}{6} \atop \frac{1}{3},\ \frac{2}{3}};\lambda y^3\,2^6 3^3\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{6s+4}\,y\,\left(\frac{(6(s+1))!}{(6(s+1)-2)!}\right)\;{}_5F_2\left[{(s+1)+\frac{1}{6},\ (s+1)+\frac{1}{3},\ (s+1)+\frac{1}{2},\ (s+1)+\frac{2}{3},\ (s+1)+\frac{5}{6} \atop \frac{2}{3},\ \frac{4}{3}};\lambda y^3\,2^6 3^3\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{6s+2}\,y^2\,\left(\frac{(6(s+1))!}{2!(6(s+1)-4)!}\right)\;{}_5F_2\left[{(s+1)+\frac{1}{6},\ (s+1)+\frac{1}{3},\ (s+1)+\frac{1}{2},\ (s+1)+\frac{2}{3},\ (s+1)+\frac{5}{6} \atop \frac{4}{3},\ \frac{5}{3}};\lambda y^3\,2^6 3^3\right]$$

$$\mathcal{H}_{7,0}(\lambda;x,y)=\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\,x^{7s}\;{}_{12}F_6\left[{\frac{s}{2}+\frac{1}{14},\ \frac{s}{2}+\frac{2}{14},\ \dots,\ \frac{s}{2}+\frac{13}{14} \atop \frac{1}{7},\ \frac{2}{7},\ \frac{3}{7},\ \frac{4}{7},\ \frac{5}{7},\ \frac{6}{7}};\lambda^2 y^7\,2^{12}7^7\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{7s+5}\,y\,\left(\frac{(7(s+1))!}{(7(s+1)-2)!}\right)\;{}_{12}F_6\left[{\frac{s+1}{2}+\frac{1}{14},\ \frac{s+1}{2}+\frac{2}{14},\ \dots,\ \frac{s+1}{2}+\frac{13}{14} \atop \frac{2}{7},\ \frac{3}{7},\ \frac{4}{7},\ \frac{5}{7},\ \frac{6}{7},\ \frac{8}{7}};\lambda^2 y^7\,2^{12}7^7\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{7s+3}\,y^2\,\left(\frac{(7(s+1))!}{(7(s+1)-4)!2!}\right)\;{}_{12}F_6\left[{\frac{s+1}{2}+\frac{1}{14},\ \frac{s+1}{2}+\frac{2}{14},\ \dots,\ \frac{s+1}{2}+\frac{13}{14} \atop \frac{3}{7},\ \frac{4}{7},\ \frac{5}{7},\ \frac{6}{7},\ \frac{8}{7},\ \frac{9}{7}};\lambda^2 y^7\,2^{12}7^7\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{7s+1}\,y^3\,\left(\frac{(7(s+1))!}{(7(s+1)-6)!3!}\right)\;{}_{12}F_6\left[{\frac{s+1}{2}+\frac{1}{14},\ \frac{s+1}{2}+\frac{2}{14},\ \dots,\ \frac{s+1}{2}+\frac{13}{14} \atop \frac{4}{7},\ \frac{5}{7},\ \frac{6}{7},\ \frac{8}{7},\ \frac{9}{7},\ \frac{10}{7}};\lambda^2 y^7\,2^{12}7^7\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+2}}{(s+2)!}\,x^{7s+6}\,y^4\,\left(\frac{(7(s+2))!}{(7(s+2)-8)!4!}\right)\;{}_{12}F_6\left[{\frac{s+2}{2}+\frac{1}{14},\ \frac{s+2}{2}+\frac{2}{14},\ \dots,\ \frac{s+2}{2}+\frac{13}{14} \atop \frac{5}{7},\ \frac{6}{7},\ \frac{8}{7},\ \frac{9}{7},\ \frac{10}{7},\ \frac{11}{7}};\lambda^2 y^7\,2^{12}7^7\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+2}}{(s+2)!}\,x^{7s+4}\,y^5\,\left(\frac{(7(s+2))!}{(7(s+2)-10)!5!}\right)\;{}_{12}F_6\left[{\frac{s+2}{2}+\frac{1}{14},\ \frac{s+2}{2}+\frac{2}{14},\ \dots,\ \frac{s+2}{2}+\frac{13}{14} \atop \frac{6}{7},\ \frac{8}{7},\ \frac{9}{7},\ \frac{10}{7},\ \frac{11}{7},\ \frac{12}{7}};\lambda^2 y^7\,2^{12}7^7\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+2}}{(s+2)!}\,x^{7s+2}\,y^6\,\left(\frac{(7(s+2))!}{(7(s+2)-12)!6!}\right)\;{}_{12}F_6\left[{\frac{s+2}{2}+\frac{1}{14},\ \frac{s+2}{2}+\frac{2}{14},\ \dots,\ \frac{s+2}{2}+\frac{13}{14} \atop \frac{8}{7},\ \frac{9}{7},\ \frac{10}{7},\ \frac{11}{7},\ \frac{12}{7},\ \frac{13}{7}};\lambda^2 y^7\,2^{12}7^7\right]$$

$$\mathcal{H}_{8,0}(\lambda;x,y)=\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\,x^{8s}\;{}_7F_3\left[{s+\frac{1}{8},\ s+\frac{1}{4},\ s+\frac{3}{8},\ s+\frac{1}{2},\ s+\frac{5}{8},\ s+\frac{3}{4},\ s+\frac{7}{8} \atop \frac{1}{4},\ \frac{1}{2},\ \frac{3}{4}};\lambda y^4\,2^{16}\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{8s+6}\,y\,\left(\frac{(8(s+1))!}{(8(s+1)-2)!}\right)\;{}_7F_3\left[{(s+1)+\frac{1}{8},\ (s+1)+\frac{1}{4},\ (s+1)+\frac{3}{8},\ (s+1)+\frac{1}{2},\ (s+1)+\frac{5}{8},\ (s+1)+\frac{3}{4},\ (s+1)+\frac{7}{8} \atop \frac{1}{2},\ \frac{3}{4},\ \frac{5}{4}};\lambda y^4\,2^{16}\right]$$

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Table 1 – continued from previous page

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\ x^{8s+4}\ y^2\ \left(\frac{(8(s+1))!}{2!(8(s+1)-4)!}\right)\ {}_7F_3\left[\begin{matrix}(s+1)+\frac{1}{8},\ (s+1)+\frac{1}{4},\ (s+1)+\frac{3}{8},\ (s+1)+\frac{1}{2},\ (s+1)+\frac{5}{8},\ (s+1)+\frac{3}{4},\ (s+1)+\frac{7}{8}\\ \frac{3}{4},\ \frac{5}{4},\ \frac{3}{2}\end{matrix};\lambda y^4\ 2^{16}\right]$$

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\ x^{8s+2}\ y^3\ \left(\frac{(8(s+1))!}{3!(8(s+1)-6)!}\right)\ {}_7F_3\left[\begin{matrix}(s+1)+\frac{1}{8},\ (s+1)+\frac{1}{4},\ (s+1)+\frac{3}{8},\ (s+1)+\frac{1}{2},\ (s+1)+\frac{5}{8},\ (s+1)+\frac{3}{4},\ (s+1)+\frac{7}{8}\\ \frac{5}{4},\ \frac{3}{2},\ \frac{7}{4}\end{matrix};\lambda y^4\ 2^{16}\right]$$

$$\mathcal{H}_{9,0}(\lambda;x,y) = \sum_{s=0}^{\infty} \frac{\lambda^s}{s!}\ x^{9s}\ {}_{16}F_8\left[\begin{matrix}\frac{s}{2}+\frac{1}{18},\ \frac{s}{2}+\frac{2}{18},\ \dots,\ \frac{s}{2}+\frac{17}{18}\\ \frac{1}{9},\ \frac{2}{9},\ \frac{1}{3},\ \frac{4}{9},\ \frac{5}{9},\ \frac{2}{3},\ \frac{7}{9},\ \frac{8}{9}\end{matrix};\lambda^2 y^9\ 2^{16}3^{18}\right]$$

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\ x^{9s+7}\ y\ \left(\frac{(9(s+1))!}{(9(s+1)-2)!}\right)\ {}_{16}F_8\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{18},\ \frac{s+1}{2}+\frac{2}{18},\ \dots,\ \frac{s+1}{2}+\frac{17}{18}\\ \frac{2}{9},\ \frac{1}{3},\ \frac{4}{9},\ \frac{5}{9},\ \frac{2}{3},\ \frac{7}{9},\ \frac{8}{9},\ \frac{10}{9}\end{matrix};\lambda^2 y^9\ 2^{16}3^{18}\right]$$

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\ x^{9s+5}\ y^2\ \left(\frac{(9(s+1))!}{(9(s+1)-4)!2!}\right)\ {}_{16}F_8\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{18},\ \frac{s+1}{2}+\frac{2}{18},\ \dots,\ \frac{s+1}{2}+\frac{17}{18}\\ \frac{1}{3},\ \frac{4}{9},\ \frac{5}{9},\ \frac{2}{3},\ \frac{7}{9},\ \frac{8}{9},\ \frac{10}{9},\ \frac{11}{9}\end{matrix};\lambda^2 y^9\ 2^{16}3^{18}\right]$$

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\ x^{9s+3}\ y^3\ \left(\frac{(9(s+1))!}{(9(s+1)-6)!3!}\right)\ {}_{16}F_8\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{18},\ \frac{s+1}{2}+\frac{2}{18},\ \dots,\ \frac{s+1}{2}+\frac{17}{18}\\ \frac{4}{9},\ \frac{5}{9},\ \frac{2}{3},\ \frac{7}{9},\ \frac{8}{9},\ \frac{10}{9},\ \frac{11}{9},\ \frac{4}{3}\end{matrix};\lambda^2 y^9\ 2^{16}3^{18}\right]$$

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\ x^{9s+1}\ y^4\ \left(\frac{(9(s+1))!}{(9(s+1)-8)!4!}\right)\ {}_{16}F_8\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{18},\ \frac{s+1}{2}+\frac{2}{18},\ \dots,\ \frac{s+1}{2}+\frac{17}{18}\\ \frac{5}{9},\ \frac{2}{3},\ \frac{7}{9},\ \frac{8}{9},\ \frac{10}{9},\ \frac{11}{9},\ \frac{4}{3},\ \frac{13}{9}\end{matrix};\lambda^2 y^9\ 2^{16}3^{18}\right]$$

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+2}}{(s+2)!}\ x^{9s+8}\ y^5\ \left(\frac{(9(s+2))!}{(9(s+2)-10)!5!}\right)\ {}_{16}F_8\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{18},\ \frac{s+2}{2}+\frac{2}{18},\ \dots,\ \frac{s+2}{2}+\frac{17}{18}\\ \frac{2}{3},\ \frac{7}{9},\ \frac{8}{9},\ \frac{10}{9},\ \frac{11}{9},\ \frac{4}{3},\ \frac{13}{9},\ \frac{14}{9}\end{matrix};\lambda^2 y^9\ 2^{16}3^{18}\right]$$

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+2}}{(s+2)!}\ x^{9s+6}\ y^6\ \left(\frac{(9(s+2))!}{(9(s+2)-12)!6!}\right)\ {}_{16}F_8\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{18},\ \frac{s+2}{2}+\frac{2}{18},\ \dots,\ \frac{s+2}{2}+\frac{17}{18}\\ \frac{7}{9},\ \frac{8}{9},\ \frac{10}{9},\ \frac{11}{9},\ \frac{4}{3},\ \frac{13}{9},\ \frac{14}{9},\ \frac{5}{3}\end{matrix};\lambda^2 y^9\ 2^{16}3^{18}\right]$$

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+2}}{(s+2)!}\ x^{9s+4}\ y^7\ \left(\frac{(9(s+2))!}{(9(s+2)-14)!7!}\right)\ {}_{16}F_8\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{18},\ \frac{s+2}{2}+\frac{2}{18},\ \dots,\ \frac{s+2}{2}+\frac{17}{18}\\ \frac{8}{9},\ \frac{10}{9},\ \frac{11}{9},\ \frac{4}{3},\ \frac{13}{9},\ \frac{14}{9},\ \frac{5}{3},\ \frac{16}{9}\end{matrix};\lambda^2 y^9\ 2^{16}3^{18}\right]$$

$$+ \sum_{s=0}^{\infty} \frac{\lambda^{s+2}}{(s+2)!}\ x^{9s+2}\ y^8\ \left(\frac{(9(s+2))!}{(9(s+2)-16)!8!}\right)\ {}_{16}F_8\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{18},\ \frac{s+2}{2}+\frac{2}{18},\ \dots,\ \frac{s+2}{2}+\frac{17}{18}\\ \frac{10}{9},\ \frac{11}{9},\ \frac{4}{3},\ \frac{13}{9},\ \frac{14}{9},\ \frac{5}{3},\ \frac{16}{9},\ \frac{17}{9}\end{matrix};\lambda^2 y^9\ 2^{16}3^{18}\right]$$

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$$\mathcal{H}_{10,0}(\lambda;x,y)=\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\,x^{10s}\;{}_9F_4\left[{s+\frac{1}{10},\,s+\frac{2}{10},\,\ldots,\,s+\frac{9}{10} \atop \frac{1}{5},\,\frac{2}{5},\,\frac{3}{5},\,\frac{4}{5}};\lambda y^5\,2^{10}5^5\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{10s+8}\,y\,\left(\frac{(10(s+1))!}{(10(s+1)-2)!}\right)\;{}_9F_4\left[{(s+1)+\frac{1}{10},\,(s+1)+\frac{2}{10},\,\ldots,\,(s+1)+\frac{9}{10} \atop \frac{2}{5},\,\frac{3}{5},\,\frac{4}{5},\,\frac{6}{5}};\lambda y^5\,2^{10}5^5\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{10s+6}\,y^2\,\left(\frac{(10(s+1))!}{2!(10(s+1)-4)!}\right)\;{}_9F_4\left[{(s+1)+\frac{1}{10},\,(s+1)+\frac{2}{10},\,\ldots,\,(s+1)+\frac{9}{10} \atop \frac{3}{5},\,\frac{4}{5},\,\frac{6}{5},\,\frac{7}{5}};\lambda y^5\,2^{10}5^5\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{10s+4}\,y^3\,\left(\frac{(10(s+1))!}{3!(10(s+1)-6)!}\right)\;{}_9F_4\left[{(s+1)+\frac{1}{10},\,(s+1)+\frac{2}{10},\,\ldots,\,(s+1)+\frac{9}{10} \atop \frac{4}{5},\,\frac{6}{5},\,\frac{7}{5},\,\frac{8}{5}};\lambda y^5\,2^{10}5^5\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\,x^{10s+2}\,y^4\,\left(\frac{(10(s+1))!}{4!(10(s+1)-8)!}\right)\;{}_9F_4\left[{(s+1)+\frac{1}{10},\,(s+1)+\frac{2}{10},\,\ldots,\,(s+1)+\frac{9}{10} \atop \frac{6}{5},\,\frac{7}{5},\,\frac{8}{5},\,\frac{9}{5}};\lambda y^5\,2^{10}5^5\right]$$

Table 2 Shifted exponential lacunary generating functions $\mathcal{H}_{K,L}(\lambda;x;y)$ for the bi-variate Hermite polynomials $H_n(x,y)$, $K=3,4$ and $L=1,2,3$.

$$(\lambda;x,y) = \sum_{s=0}^{\infty} \frac{\lambda^s}{s!}\ \left(x^{3s+1} + x^{3s-1}\binom{3s}{1}2y\right)\ {}_4F_2\left[\begin{matrix}\frac{s}{2}+\frac{1}{6},\ \frac{s}{2}+\frac{1}{3},\ \frac{s}{2}+\frac{2}{3},\ \frac{s}{2}+\frac{5}{6}\\ \frac{1}{3},\ \frac{2}{3}\end{matrix};\lambda^2 y^3\ 2^4 3^3\right]$$

$$+\sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\ \left(x^{3(s+1)-1}y + x^{3(s+1)-3}\binom{3(s+1)-2}{1}2y^2\right)\ \left(\frac{(3(s+1))!}{(3(s+1)-2)!}\right)\ {}_4F_2\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{6},\ \frac{s+1}{2}+\frac{1}{3},\ \frac{s+1}{2}+\frac{2}{3},\ \frac{s+1}{2}+\frac{5}{6}\\ \frac{2}{3},\ \frac{4}{3}\end{matrix};\lambda^2 y^3\ 2^4 3^3\right]$$

$$+\sum_{s=0}^{\infty} \frac{\lambda^{s+2}}{(s+2)!}\ \left(x^{3(s+2)-3}y^2 + x^{3(s+2)-5}\binom{3(s+2)-4}{1}2y^3\right)\ \left(\frac{(3(s+2))!}{(3(s+2)-4)!2!}\right)\ {}_4F_2\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{6},\ \frac{s+2}{2}+\frac{1}{3},\ \frac{s+2}{2}+\frac{2}{3},\ \frac{s+2}{2}+\frac{5}{6}\\ \frac{4}{3},\ \frac{5}{3}\end{matrix};\lambda^2 y^3\ 2^4 3^3\right]$$

$$(\lambda;x,y) = \sum_{s=0}^{\infty} \frac{\lambda^s}{s!}\ \left(x^{3s}\left(x^2+2y\right) + x^{3s}\binom{3s}{1}4y + x^{3s-2}\binom{3s}{2}2!\cdot 4y^2\right)\ {}_4F_2\left[\begin{matrix}\frac{s}{2}+\frac{1}{6},\ \frac{s}{2}+\frac{1}{3},\ \frac{s}{2}+\frac{2}{3},\ \frac{s}{2}+\frac{5}{6}\\ \frac{1}{3},\ \frac{2}{3}\end{matrix};\lambda^2 y^3\ 2^4 3^3\right]$$

$$+\sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\ \left(x^{3(s+1)-2}y\left(x^2+2y\right) + x^{3(s+1)-2}\binom{3(s+1)-2}{1}4y^2 + x^{3(s+1)-4}\binom{3(s+1)-2}{2}2!\cdot 4y^3\right)\ \left(\frac{(3(s+1))!}{(3(s+1)-2)!}\right)\cdot$$
$$\cdot\ {}_4F_2\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{6},\ \frac{s+1}{2}+\frac{1}{3},\ \frac{s+1}{2}+\frac{2}{3},\ \frac{s+1}{2}+\frac{5}{6}\\ \frac{2}{3},\ \frac{4}{3}\end{matrix};\lambda^2 y^3\ 2^4 3^3\right]$$

$$+\sum_{s=0}^{\infty} \frac{\lambda^{s+2}}{(s+2)!}\ \left(x^{3(s+2)-4}y^2\left(x^2+2y\right) + x^{3(s+2)-4}\binom{3(s+2)-4}{1}4y^3 + x^{3(s+2)-6}\binom{3(s+2)-4}{2}2!\cdot 4y^4\right)\ \left(\frac{(3(s+2))!}{(3(s+2)-4)!2!}\right)\cdot$$
$$\cdot\ {}_4F_2\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{6},\ \frac{s+2}{2}+\frac{1}{3},\ \frac{s+2}{2}+\frac{2}{3},\ \frac{s+2}{2}+\frac{5}{6}\\ \frac{4}{3},\ \frac{5}{3}\end{matrix};\lambda^2 y^3\ 2^4 3^3\right]$$

$$(\lambda;x,y) = \sum_{s=0}^{\infty} \frac{\lambda^s}{s!}\ \left(x^{3s}\left(x^3+6xy\right) + x^{3s-1}\binom{3s}{1}6y\left(x^2+2y\right) + x^{3s-1}\binom{3s}{2}2!\cdot 12y^2 + x^{3s-3}\binom{3s}{3}3!\cdot 8y^3\right)\ {}_4F_2\left[\begin{matrix}\frac{s}{2}+\frac{1}{6},\ \frac{s}{2}+\frac{1}{3},\ \frac{s}{2}+\frac{2}{3},\ \frac{s}{2}+\frac{5}{6}\\ \frac{1}{3},\ \frac{2}{3}\end{matrix};\lambda^2 y^3\ 2^4 3^3\right]$$

$$+\sum_{s=0}^{\infty} \frac{\lambda^{s+1}}{(s+1)!}\ \left(x^{3(s+1)-2}y\left(x^3+6xy\right) + x^{3(s+1)-3}\binom{3(s+1)-2}{1}6y^2\left(x^2+2y\right) + x^{3(s+1)-3}\binom{3(s+1)-2}{2}2!\cdot 12y^3\right.$$

$$\left.+x^{3(s+1)-5}\binom{3(s+1)-2}{3}3!\cdot 8y^4\right)\ \left(\frac{(3(s+1))!}{(3(s+1)-2)!}\right)\ {}_4F_2\left[\begin{matrix}\frac{s+1}{2}+\frac{1}{6},\ \frac{s+1}{2}+\frac{1}{3},\ \frac{s+1}{2}+\frac{2}{3},\ \frac{s+1}{2}+\frac{5}{6}\\ \frac{2}{3},\ \frac{4}{3}\end{matrix};\lambda^2 y^3\ 2^4 3^3\right]$$

$$+\sum_{s=0}^{\infty} \frac{\lambda^{s+2}}{(s+2)!}\ \left(x^{3(s+2)-4}y^2\left(x^3+6xy\right) + x^{3(s+2)-5}\binom{3(s+2)-4}{1}6y^3\left(x^2+2y\right) + x^{3(s+2)-5}\binom{3(s+2)-4}{2}2!\cdot 12y^4\right.$$
$$\left.+x^{3(s+2)-7}\binom{3(s+2)-4}{3}3!\cdot 8y^5\right)\ \left(\frac{(3(s+2))!}{(3(s+2)-4)!2!}\right)\ {}_4F_2\left[\begin{matrix}\frac{s+2}{2}+\frac{1}{6},\ \frac{s+2}{2}+\frac{1}{3},\ \frac{s+2}{2}+\frac{2}{3},\ \frac{s+2}{2}+\frac{5}{6}\\ \frac{4}{3},\ \frac{5}{3}\end{matrix};\lambda^2 y^3\ 2^4 3^3\right]$$

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$$(\lambda;x,y)=\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\ \left(x^{4s+1}+x^{4s-1}\binom{4s}{1}2y\right)\ {}_3F_1\left[\begin{matrix}s+\frac14,\ s+\frac12,\ s+\frac34\\ \frac12\end{matrix};\lambda y^2\,2^6\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\ \left(x^{4s+3}y+x^{4s+1}\binom{4s+2}{1}2y^2\right)\ \left(\frac{(4(s+1))!}{(4(s+1)-2)!}\right)\ {}_3F_1\left[\begin{matrix}(s+1)+\frac14,\ (s+1)+\frac12,\ (s+1)+\frac34\\ \frac32\end{matrix};\lambda y^2\,2^6\right]$$

$$(\lambda;x,y)=\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\ \left(x^{4s}\left(x^2+2y\right)+x^{4s}\binom{4s}{1}4y+x^{4s-2}\binom{4s}{2}2!\cdot 4y^2\right)\ {}_3F_1\left[\begin{matrix}s+\frac14,\ s+\frac12,\ s+\frac34\\ \frac12\end{matrix};\lambda y^2\,2^6\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\ \left(x^{4s+2}y\left(x^2+2y\right)+x^{4s+2}\binom{4s+2}{1}4y^2+x^{4s}\binom{4s+2}{2}2!\cdot 4y^3\right)\ \left(\frac{(4(s+1))!}{(4(s+1)-2)!}\right)\ {}_3F_1\left[\begin{matrix}(s+1)+\frac14,\ (s+1)+\frac12,\ (s+1)+\frac34\\ \frac32\end{matrix};\lambda y^2\,2^6\right]$$

$$(\lambda;x,y)=\sum_{s=0}^{\infty}\frac{\lambda^s}{s!}\ \left(x^{4s}\left(x^3+6xy\right)+x^{4s-1}\binom{4s}{1}6y\left(x^2+2y\right)+x^{4s-1}\binom{4s}{2}2!\cdot 12y^2+x^{4s-3}\binom{4s}{3}3!\cdot 8y^3\right)\ {}_3F_1\left[\begin{matrix}s+\frac14,\ s+\frac12,\ s+\frac34\\ \frac12\end{matrix};\lambda y^2\,2^6\right]$$

$$+\sum_{s=0}^{\infty}\frac{\lambda^{s+1}}{(s+1)!}\ \left(x^{4s+2}y\left(x^3+6xy\right)+x^{4s+1}\binom{4s+2}{1}6y^2\left(x^2+2y\right)+x^{4s+1}\binom{4s+2}{2}2!\cdot 12y^3+x^{4s-1}\binom{4s+2}{3}3!\cdot 8y^4\right)\ \cdot$$
$$\cdot\ \left(\frac{(4(s+1))!}{(4(s+1)-2)!}\right)\ {}_3F_1\left[\begin{matrix}(s+1)+\frac14,\ (s+1)+\frac12,\ (s+1)+\frac34\\ \frac32\end{matrix};\lambda y^2\,2^6\right]$$

Conclusion and outlook

Conclusion and outlook

- using some elementary techniques such as re-sorting of formal power series, we obtained **all higher order exponential lacunary generating functions** for the Hermite polynomials $H_n(x, y)$
- since the polynomials $H_n(x, y)$ are related to a large number of other (Appell-type) families of polynomials, we expect our results to be generalizable to these families
- the range of possibilities is even further extended via so-called **umbral image techniques** – joint work with G. Dattoli (ENEA, Rome), G.H.E. Duchamp (Paris 13), S. Licciardi (ENEA, Rome) and K.A. Penson

Last October at IHÉS. . .

Thank you!

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