Tracelets

And Tracelet Analysis for Compositional Rewriting Systems

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Partially based upon previous work with:
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Vincent Danos and **Ilias Garnier (ENS Paris)**

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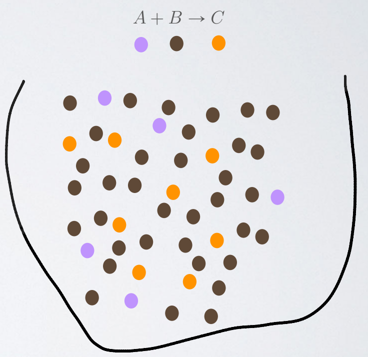
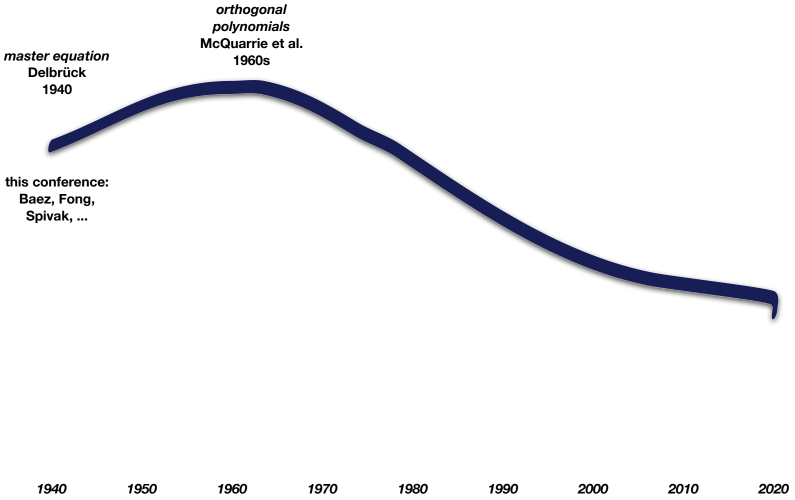
Chemical reaction systems

- **State:** a pool of **indistinguishable particles** (of different **types**)
- **Transition:** e.g. $A + B \longrightarrow C$
  - (i) **select at random** a type A and a type B particle; **remove these**
  - (ii) **add** a particle of type C
- **Dynamics:** transitions occur **at random** with probability proportional to **number of possibilities** that the **input pattern** may be found in a state

$\Rightarrow$ **highly** intricate stochastic dynamics!



Historical overview (rough sketch)



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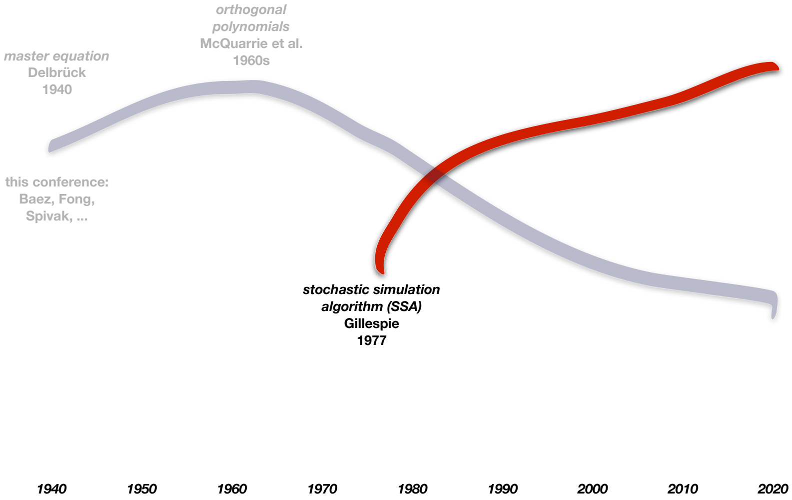
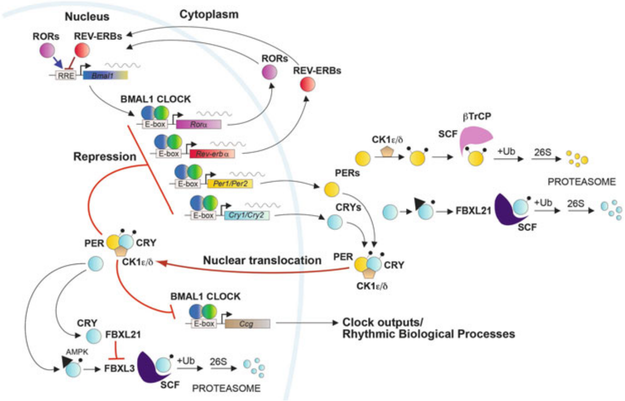


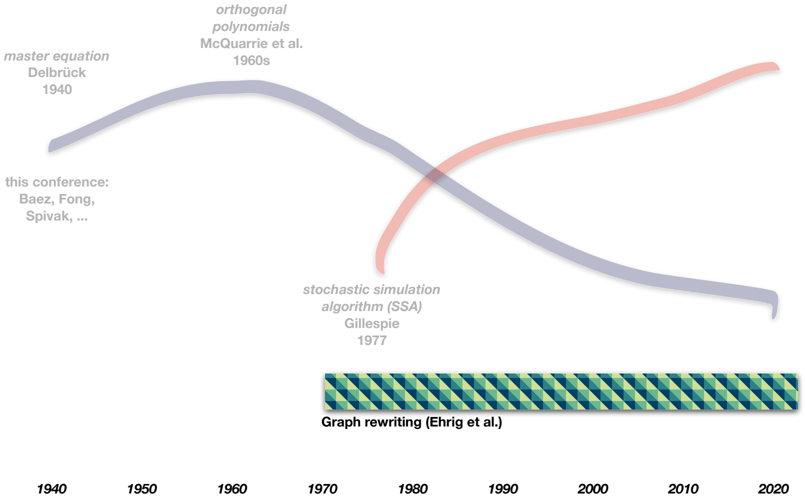
Modern systems biology: pathways



Model of the circadian clock in mammals. (*source:* [1])

[1] "A Time for Metabolism and Hormones". In: *Research and Perspectives in Endocrine Interactions* (2016)

Historical overview (rough sketch)



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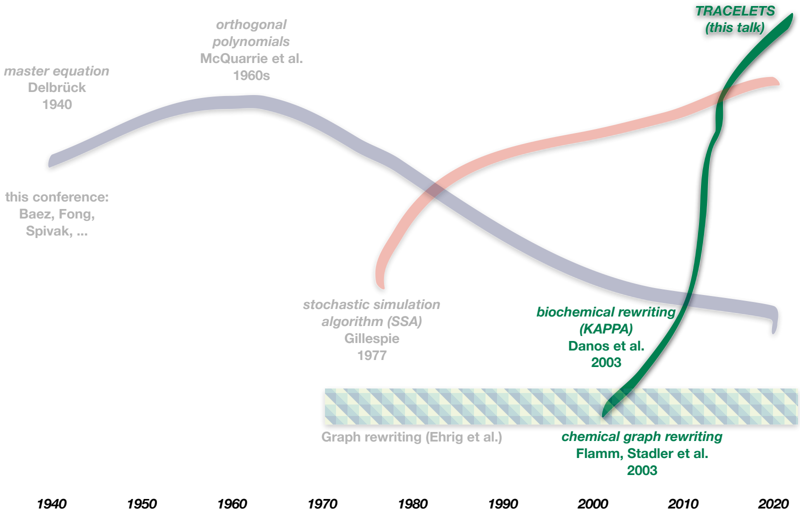


The basic setup for compositional rewriting

Adhesive and extensive categories (cf. [2], Def. 3.1 ff)

A category $\mathbf{C}$ is said to be **adhesive** if

(i) $\mathbf{C}$ has pushouts along **monomorphisms**,

(ii) $\mathbf{C}$ has pullbacks, and if

(iii) pushouts along monomorphisms are **van Kampen (VK) squares**.

If $\mathbf{C}$ in addition possesses a **strict initial object** $\varnothing \in ob(\mathbf{C})$, i.e. an object s.th. $\forall X \in ob(\mathbf{C}) : \exists! i_X : \varnothing \hookrightarrow X$ and all $X \to \varnothing$ are isos, the category is said to be **extensive**. It is called **finitary** [3] if every object X has only finitely many subobjects (up to iso).

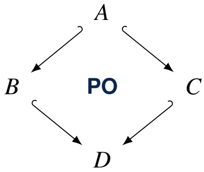
- **Examples for finitary adhesive extensive categories [3]:**
  - **FinSet**, the category of **(finite) sets and set functions**
  - **FinGraph**, the category of **(finite) directed multigraphs and graph homomorphisms** (and also **colored/typed graphs, attributed graphs, hypergraphs,...**)
  - different variants of categories of finite **typed** or **attributed graphs** (Kappa!)

[2] Stephen Lack and Paweł Sobociński. “Adhesive and quasiadhesive categories”. In: *RAIRO-Theoretical Informatics and Applications* 39.3 (2005), pp. 511–545

[3] Karsten Gabriel et al. “Finitary $\mathscr{M}$-adhesive categories”. In: *Mathematical Structures in Computer Science* 24.04 (June 2014)

Brief comments on abstract category-theoretical operations:

- **pushout (PO) along monomorphisms** in the category **Set**:



Interpretation:

A	–	**intersection** of B and C in D
D	–	**union** of B and C along A

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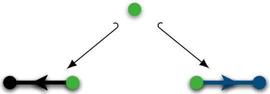

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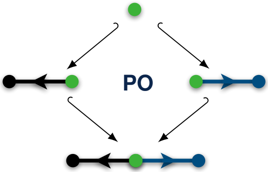

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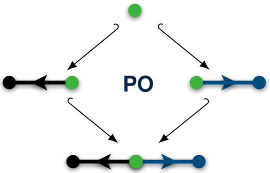

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- **pushout complement (POC)** of $D \hookleftarrow B \hookleftarrow A$: a set C and monomorphisms $D \hookleftarrow C \hookleftarrow A$ such that the square $\square(ABDC)$ is a **pushout**

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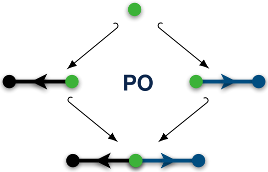

Interpretation:

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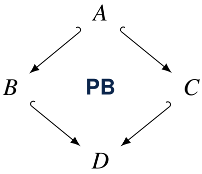
- **pushout complement (POC)** of $D \hookleftarrow B \hookleftarrow A$: a set C and monomorphisms $D \hookleftarrow C \hookleftarrow A$ such that the square $\square(ABDC)$ is a **pushout**

- **pullback (PB) along monomorphisms** in the category $\mathbf{Set}$:



Interpretation: A — **intersection** of B and C in D

Double-Pushout (DPO), $DPO^{\dagger}$ and Sesqui-Pushout (SqPO) rewriting

$$\mathbf{Lin}(\mathbf{C}) := \left\{ O \xleftarrow{o} K \xrightarrow{i} I \,\middle|\, o,i \in \mathbf{mono}(\mathbf{C}) \right\} \diagup_{\cong}$$

A **rule application** of a rule $r \in \mathbf{Lin}(\mathbf{C})$ to an object X along a **$\mathbb{T}$-admissible match** m (resp. m^* for $DPO^{\dagger}$) is defined via the following type of commutative diagram (referred to as a **direct derivation** in the literature):

$$\begin{array}{ccc} O & \xleftharpoondown{\;r\;} & I \\ {\scriptstyle m^*}\downarrow & \mathbb{T} & \downarrow{\scriptstyle m} \\ r_m(X) & \Longleftarrow & X \end{array} \quad := \quad \begin{array}{ccccc} O & \xleftarrow{\;o\;} & K & \xrightarrow{\;i\;} & I \\ {\scriptstyle m^*}\downarrow & (B) & \downarrow & (A) & \downarrow{\scriptstyle m} \\ r_m(X) & \leftarrow & \overline{K} & \rightarrow & X \end{array}$$

The precise details and **$\mathbb{T}$-type admissibility** are defined via

Type $\mathbb{T}$	nature of (B)	nature of (A)
DPO	**PO**	**POC**
$DPO^{\dagger}$	**POC**	**PO**
$SqPO$	**PO**	**FPC**

where **POC** indicates that these POCs must be constructible for admissible matches.

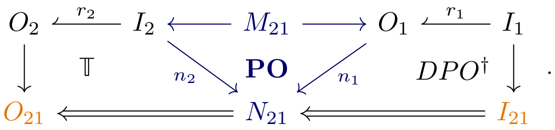
Key operation: rule compositions [4], [5]

Set of $\mathbb{T}$-type admissible matches of r_2 into r_1 for $\mathbb{T} \in \{DPO, SqPO\}$:

$$\mathbf{M}^{\mathbb{T}}_{r_2}(r_1) := \Big\{\mu_{21} = (I_2 \leftarrow M_{21} \rightarrow O_2) \Big| n_1, n_2 \text{ in } \mathbf{PO}(\mu_{21}) = (I_2 \xrightarrow{n_2} N_{21} \xleftarrow{n_1} O_1)$$

$$\text{satisfy } n_2 \in \mathbf{M}^{\mathbb{T}}_{r_2}(N_{21}) \ \wedge \ n_1 \in \mathbf{M}^{DPO^\dagger}_{r_1}(N_{21})\Big\}\,.$$

For a $\mathbb{T}$-type admissible match $\mu_{21} = (I_2 \leftarrow M_{21} \rightarrow O_2) \in \mathbf{M}^{\mathbb{T}}_{r_2}(r_1)$, construct

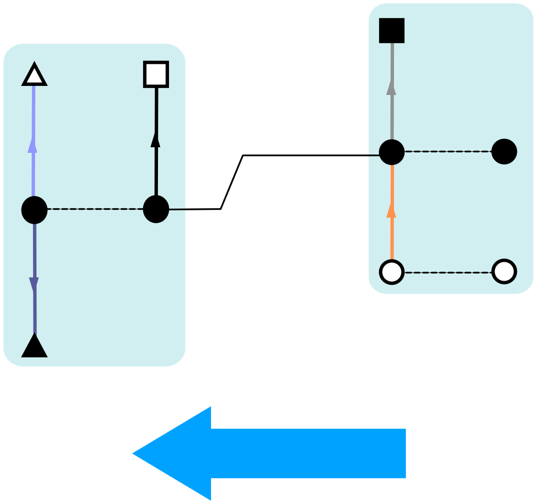


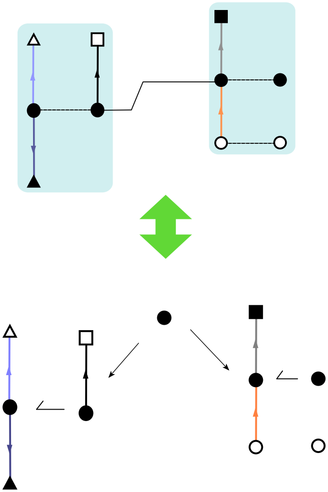
From this diagram, one may compute (via pullback composition $\circ$ of the two composable spans in the bottom row) a span of monomorphisms $(O_{21} \Leftarrow I_{21}) \in \mathbf{Lin}(\mathbf{C})$, which we define to be the $\mathbb{T}$*-type composition of* r_2 *with* r_1 *along* μ_{21} (for $\mathbb{T} \in \{DPO, SqPO\}$ as in (9)):

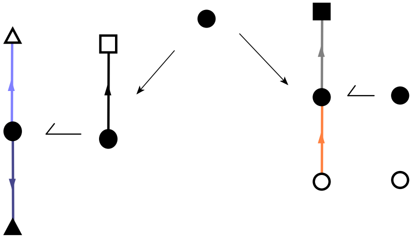
$$r_2{}^{\mu_{21}}\!\lhd_{\mathbb{T}}\, r_1 := (O_{21} \Leftarrow I_{21}) = (O_{21} \Leftarrow N_{21}) \circ (N_{21} \Leftarrow I_{21})\,.$$

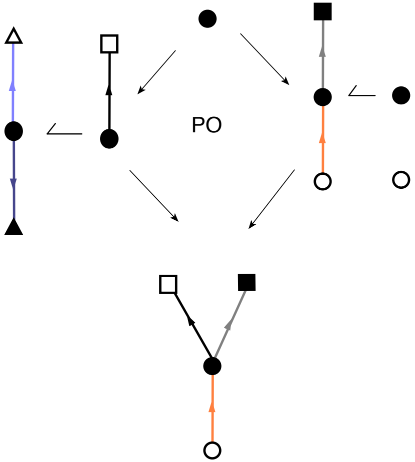
[4] Nicolas Behr and Pawel Sobocinski. "Rule Algebras for Adhesive Categories". In: *27th EACSL Annual Conference on Computer Science Logic (CSL 2018)*. Ed. by Dan Ghica and Achim Jung. Vol. 119. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Sept. 2018, 11:1–11:21

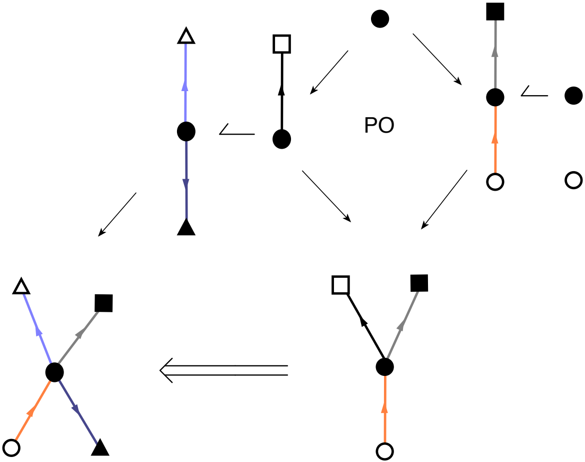
[5] Nicolas Behr. "Sesqui-Pushout Rewriting: Concurrency, Associativity and Rule Algebra Framework". In: *arXiv preprint 1904.08357* (2019)








PO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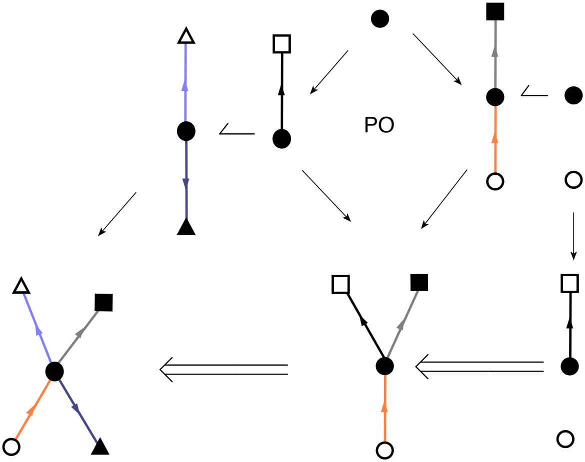
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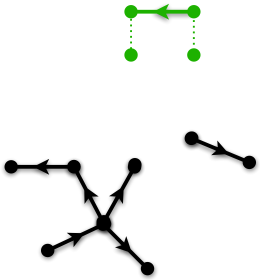
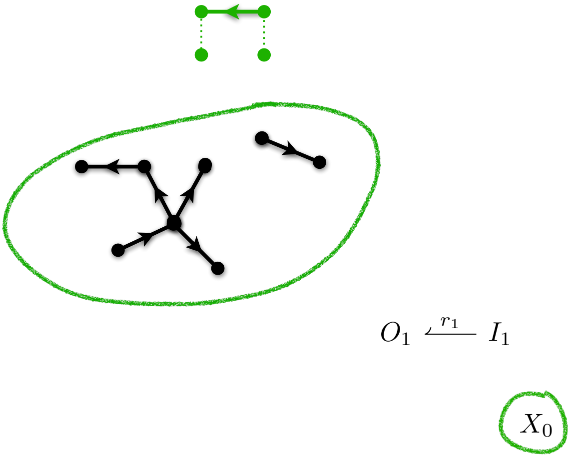


$O_1 \xleftarrow{r_1} I_1$

X_0



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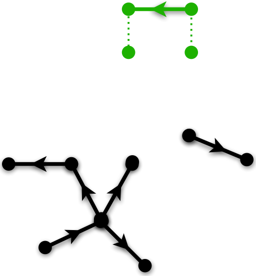
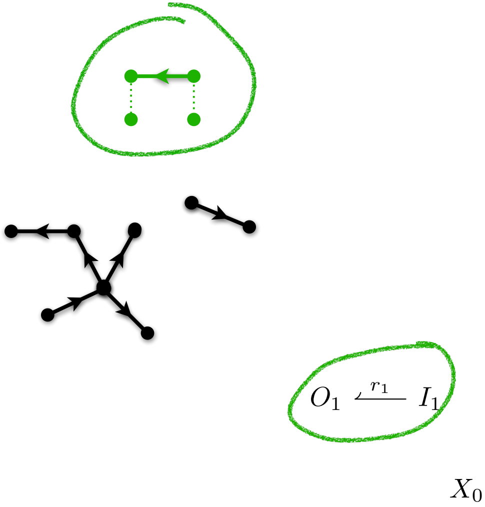


$$O_1 \xhookleftarrow{r_1} I_1$$

$$X_0$$



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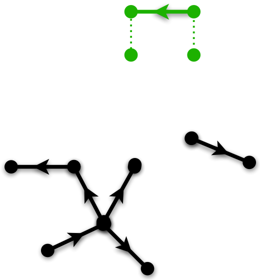
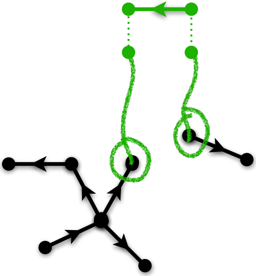
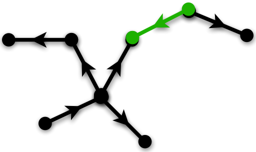


$$O_1 \xhookleftarrow{r_1} I_1$$

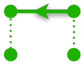
$$X_0$$

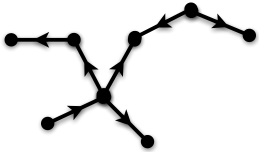


$$O_1 \xleftarrow{\;r_1\;} I_1 \xrightarrow{\;m_1\;} X_0$$



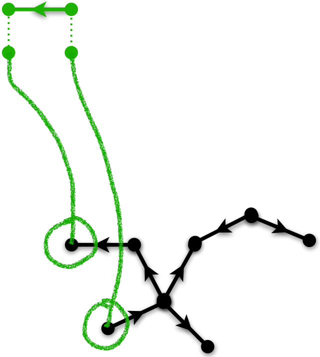
$$\begin{array}{ccc} O_1 & \xleftarrow{r_1} & I_1 \\ {\scriptstyle m_1^*}\downarrow & & \downarrow{\scriptstyle m_1} \\ X_1 & \overset{}{\underset{r_1, m_1}{\Longleftarrow}} & X_0 \end{array}$$



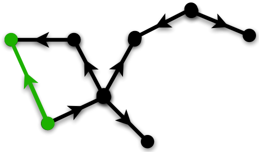


$$O_2 \xleftarrow{\;r_2\;} I_2 \qquad O_1 \xleftarrow{\;r_1\;} I_1$$

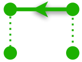
$$\begin{array}{ccc} O_1 & \xleftarrow{\;r_1\;} & I_1 \\ \downarrow{\scriptstyle m_1^*} & & \downarrow{\scriptstyle m_1} \\ X_1 & \overset{}{\underset{r_1,m_1}{\Longleftarrow}} & X_0 \end{array}$$

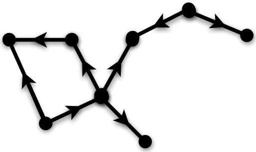


$$\begin{array}{ccccccc} O_2 & \xleftarrow{r_2} & I_2 & & O_1 & \xleftarrow{r_1} & I_1 \\ & & \searrow^{m_2} & & \swarrow_{m_1^*} & & \downarrow^{m_1} \\ & & & X_1 & \xLeftarrow[r_1,m_1]{} & & X_0 \end{array}$$

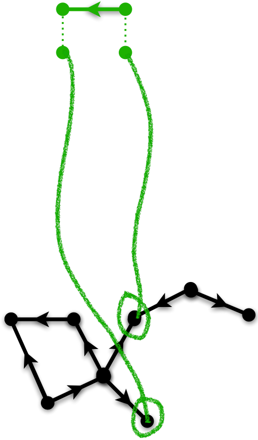


$$
\begin{array}{ccccccc}
O_2 & \xhookleftarrow{r_2} & I_2 & & O_1 & \xhookleftarrow{r_1} & I_1 \\
\downarrow{\scriptstyle m_2^*} & & & {\scriptstyle m_2}\searrow \quad \swarrow{\scriptstyle m_1^*} & & & {\scriptstyle m_1}\searrow \\
X_2 & & \overset{}{\underset{r_2, m_2}{\Longleftarrow}} & X_1 & & \underset{r_1, m_1}{\Longleftarrow} & X_0
\end{array}
$$

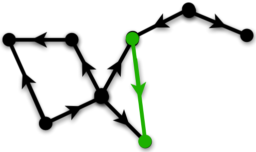




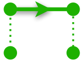
$$\begin{array}{ccccccc} & O_2 & \xleftarrow{r_2} & I_2 & O_1 & \xleftarrow{r_1} & I_1 \\ & \swarrow m_2^* & & m_2 \searrow \;\; \swarrow m_1^* & & & m_1 \searrow \\ \cdots & X_2 & \overset{}{\underset{r_2,m_2}{\Longleftarrow}} & X_1 & \underset{r_1,m_1}{\Longleftarrow} & & X_0 \end{array}$$

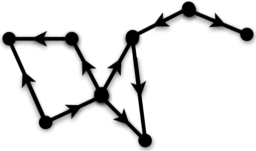


$$
\begin{array}{ccccccc}
 & O_2 & \overset{r_2}{\hookleftarrow} & I_2 & O_1 & \overset{r_1}{\hookleftarrow} & I_1 \\
 & \downarrow m_2^* & & m_2 \searrow & \swarrow m_1^* & & m_1 \searrow \\
\cdots & X_2 & \underset{r_2,m_2}{\Longleftarrow} & & X_1 & \underset{r_1,m_1}{\Longleftarrow} & X_0
\end{array}
$$

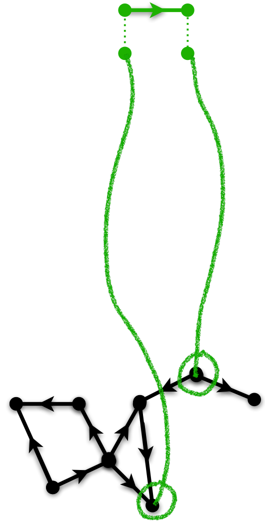


$$\begin{array}{ccccccc}
 & O_2 & \xleftarrow{\;r_2\;} & I_2 & O_1 & \xleftarrow{\;r_1\;} & I_1 \\
 & \downarrow{\scriptstyle m_2^*} & & {\scriptstyle m_2}\searrow \quad \swarrow{\scriptstyle m_1^*} & & & {\scriptstyle m_1}\searrow \\
\cdots & X_2 & \overset{}{\underset{r_2, m_2}{\Longleftarrow}} & X_1 & & \underset{r_1, m_1}{\Longleftarrow} & X_0
\end{array}$$

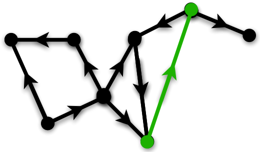




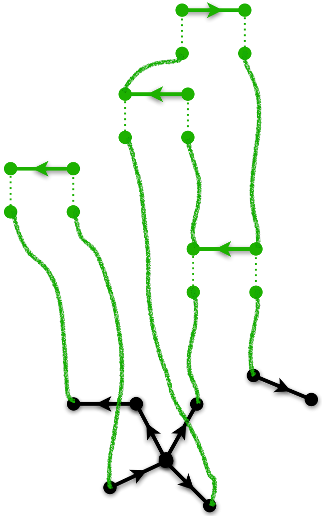
$$\begin{array}{ccccccccccc}
O_n & \overset{r_n}{\hookleftarrow} & I_n & & & & O_2 & \overset{r_2}{\hookleftarrow} & I_2 & & O_1 & \overset{r_1}{\hookleftarrow} & I_1 \\
 & & & & & & \downarrow{\scriptstyle m_2^*} & & {\scriptstyle m_2}\searrow & & \swarrow{\scriptstyle m_1^*} & & {\scriptstyle m_1}\searrow \\
 & & & X_{n-1} & & \cdots & X_2 & & \underset{r_2,m_2}{\Longleftarrow} & X_1 & & \underset{r_1,m_1}{\Longleftarrow} & X_0
\end{array}$$



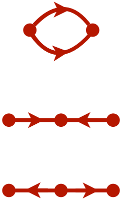
$$
\begin{array}{ccccccccccccc}
O_n & \hookleftarrow^{r_n} & I_n & & & & O_2 & \hookleftarrow^{r_2} & I_2 & & O_1 & \hookleftarrow^{r_1} & I_1 \\
 & & & \searrow^{m_n} & & & \swarrow^{m_2^*} & & & \searrow^{m_2} \ \swarrow^{m_1^*} & & & \searrow^{m_1} \\
 & & & & X_{n-1} & \cdots & X_2 & & \overset{}{\underset{r_2, m_2}{\Longleftarrow}} & X_1 & \underset{r_1, m_1}{\Longleftarrow} & & X_0
\end{array}
$$

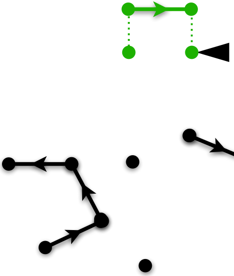


$$
\begin{array}{ccccccccccc}
 & O_n & \xhookleftarrow{r_n} & I_n & & & O_2 & \xhookleftarrow{r_2} & I_2 \quad O_1 & \xhookleftarrow{r_1} & I_1 \\
 & \downarrow{\scriptstyle m_n^*} & & {\scriptstyle m_n}\searrow & & & \downarrow{\scriptstyle m_2^*} & & {\scriptstyle m_2}\searrow \; \swarrow{\scriptstyle m_1^*} & & {\scriptstyle m_1}\searrow \\
X_n & & \xLeftarrow[r_n, m_n]{} & & X_{n-1} & \cdots & X_2 & \xLeftarrow[r_2, m_2]{} & X_1 & \xLeftarrow[r_1, m_1]{} & X_0
\end{array}
$$



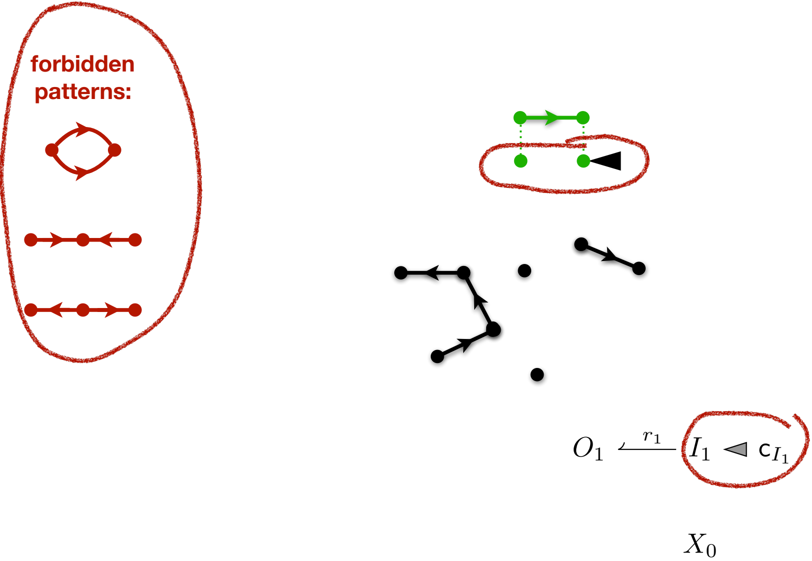
forbidden patterns:



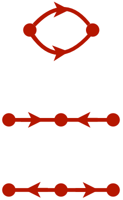


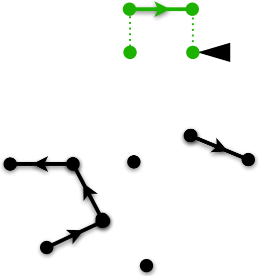
$$O_1 \xhookleftarrow{r_1} I_1 \lhd \mathsf{c}_{I_1}$$

$$X_0$$


forbidden patterns:
$O_1 \xleftarrow{r_1} I_1 \lhd \mathsf{c}_{I_1}$
X_0

forbidden patterns:

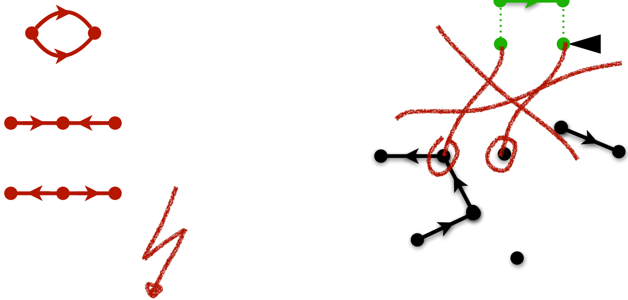




$$O_1 \xhookleftarrow{r_1} I_1 \triangleleft \mathsf{c}_{I_1}$$

$$X_0$$

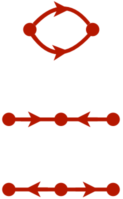
forbidden patterns:

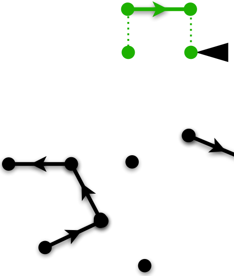


$$O_1 \xleftarrow{r_1} I_1 \lhd \mathsf{c}_{I_1}$$

X_0

forbidden patterns:

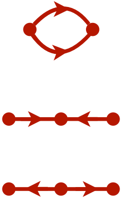


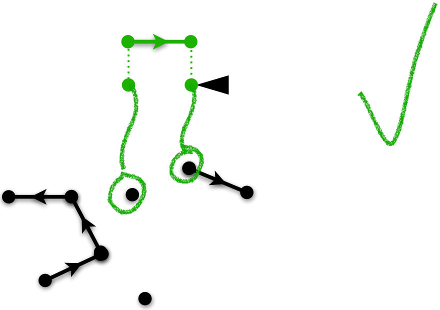


$$O_1 \xhookleftarrow{r_1} I_1 \lhd \mathsf{c}_{I_1}$$

$$X_0$$

forbidden patterns:

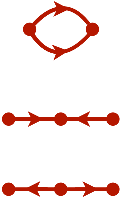


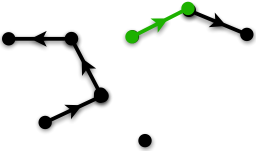


$$O_1 \xhookleftarrow{r_1} I_1 \triangleleft \mathsf{c}_{I_1}$$

$$X_0$$

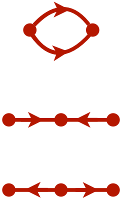
forbidden patterns:

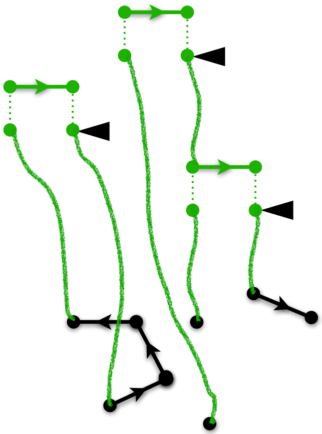


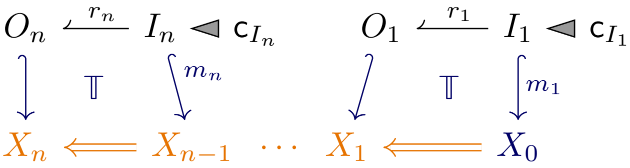


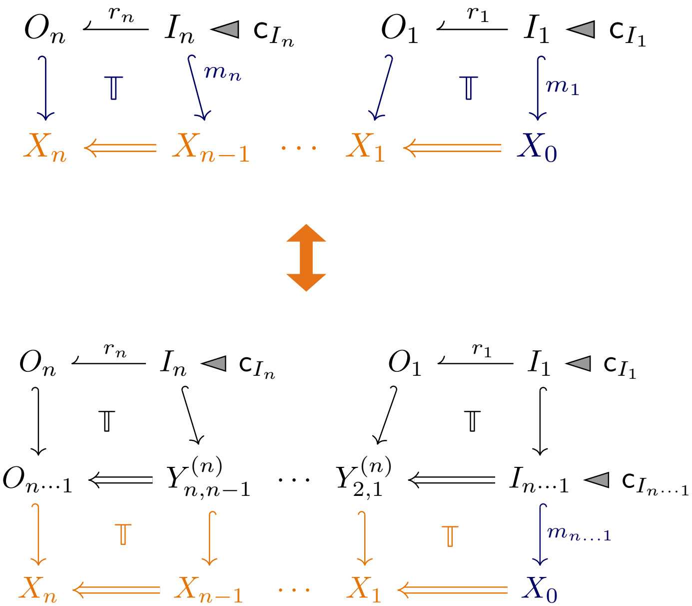
$$\begin{array}{ccccc} O_1 & \xleftarrow{r_1} & I_1 & \lhd & \mathsf{c}_{I_1} \\ \downarrow & & \downarrow & & \\ X_1 & \Longleftarrow & X_0 & & \end{array}$$

forbidden patterns:

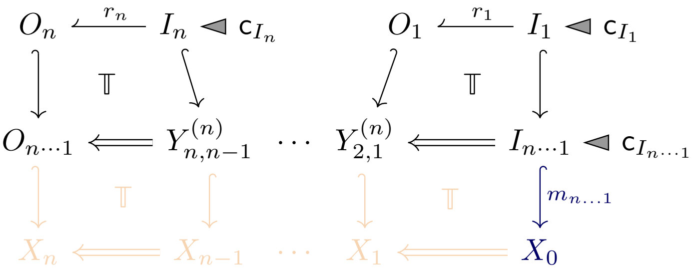




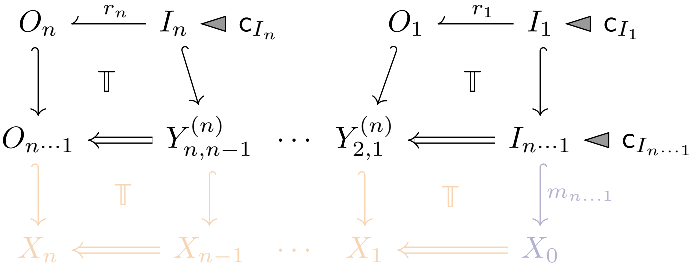

O_n
r_n
I_n
c_{I_n}
O_1
r_1
I_1
c_{I_1}
$\mathbb{T}$
m_n
$\mathbb{T}$
m_1
X_n
X_{n-1}
$\cdots$
X_1
X_0

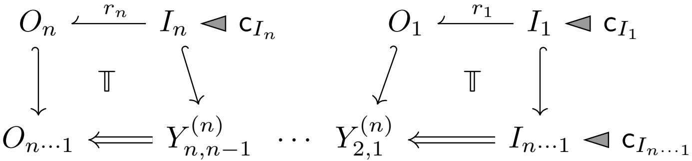


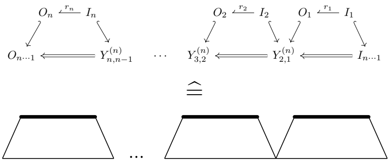
O_n r_n I_n c_{I_n} O_1 r_1 I_1 c_{I_1}
$\mathbb{T}$ m_n $\mathbb{T}$ m_1
X_n X_{n-1} $\cdots$ X_1 X_0
O_n r_n I_n c_{I_n} O_1 r_1 I_1 c_{I_1}
$\mathbb{T}$ $\mathbb{T}$
$O_{n\cdots 1}$ $Y^{(n)}_{n,n-1}$ $\cdots$ $Y^{(n)}_{2,1}$ $I_{n\cdots 1}$ $\mathsf{c}_{I_{n\cdots 1}}$
$\mathbb{T}$ $\mathbb{T}$ $m_{n\ldots 1}$
X_n X_{n-1} $\cdots$ X_1 X_0



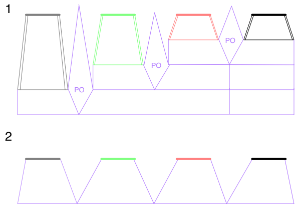
O_n
r_n
I_n
c_{I_n}
O_1
r_1
I_1
c_{I_1}
$\mathbb{T}$
$\mathbb{T}$
$O_{n\cdots 1}$
$Y^{(n)}_{n,n-1}$
$\cdots$
$Y^{(n)}_{2,1}$
$I_{n\cdots 1}$
$\mathsf{c}_{I_{n\cdots 1}}$
$\mathbb{T}$
$\mathbb{T}$
$m_{n\ldots 1}$
X_n
X_{n-1}
$\cdots$
X_1
X_0


O_n
r_n
I_n
c_{I_n}
O_1
r_1
I_1
c_{I_1}
$\mathbb{T}$
$\mathbb{T}$
$O_{n\cdots 1}$
$Y^{(n)}_{n,n-1}$
$\cdots$
$Y^{(n)}_{2,1}$
$I_{n\cdots 1}$
$\mathsf{c}_{I_{n\cdots 1}}$
$\mathbb{T}$
$\mathbb{T}$
$m_{n\ldots 1}$
X_n
X_{n-1}
$\cdots$
X_1
X_0

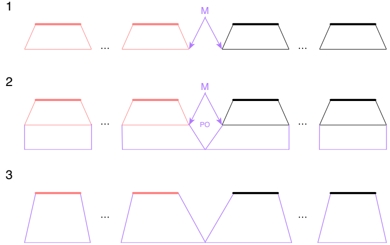

O_n
r_n
$I_n \lhd \mathsf{c}_{I_n}$
O_1
r_1
$I_1 \lhd \mathsf{c}_{I_1}$
$\mathbb{T}$
$\mathbb{T}$
$O_{n\cdots 1}$
$Y^{(n)}_{n,n-1}$
$\cdots$
$Y^{(n)}_{2,1}$
$I_{n\cdots 1} \lhd \mathsf{c}_{I_{n\cdots 1}}$



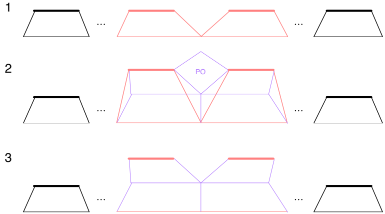
(a) Tracelets as (minimal) derivation traces.



(b) Tracelet generation (Definition 2.1).



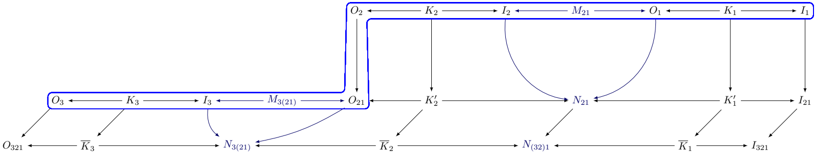
(c) Tracelet composition (Definition 2.2).



(d) Tracelet analysis (Section 3).

Figure 2 Schematic overview of the tracelet and tracelet analysis framework.

Key property: compositional associativity [6], [7], [8]

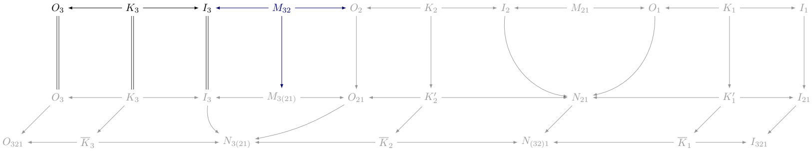


[6] Nicolas Behr and Pawel Sobocinski. "Rule Algebras for Adhesive Categories". In: *27th EACSL Annual Conference on Computer Science Logic (CSL 2018)*. Ed. by Dan Ghica and Achim Jung. Vol. 119. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Sept. 2018, 11:1–11:21

[7] Nicolas Behr. "Sesqui-Pushout Rewriting: Concurrency, Associativity and Rule Algebra Framework". In: *arXiv preprint 1904.08357* (2019)

[8] Nicolas Behr and Jean Krivine. "Compositionality of Rewriting Rules with Conditions". In: *arXiv preprint 1904.09322* (2019)

Key property: compositional associativity [6], [7], [8]



[6] Nicolas Behr and Pawel Sobocinski. “Rule Algebras for Adhesive Categories”. In: *27th EACSL Annual Conference on Computer Science Logic (CSL 2018)*. Ed. by Dan Ghica and Achim Jung. Vol. 119. Leibniz International Proceedings in Informatics (LIPIcs). Dagstuhl, Germany: Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik, Sept. 2018, 11:1–11:21

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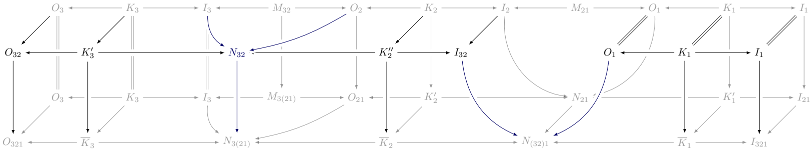
Key property: compositional associativity [6], [7], [8]



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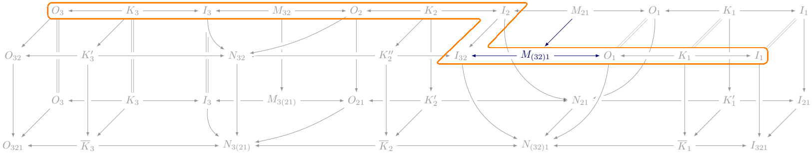
Key property: compositional associativity [6], [7], [8]



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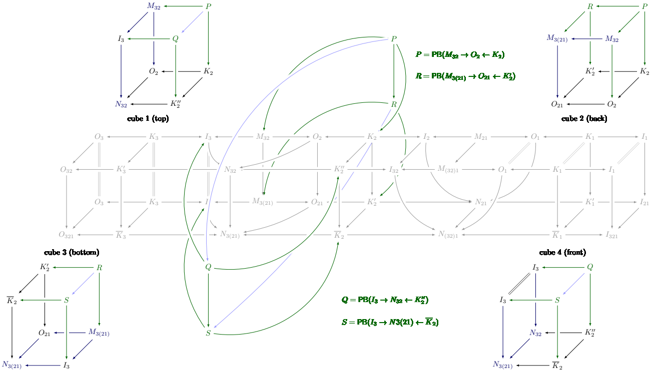

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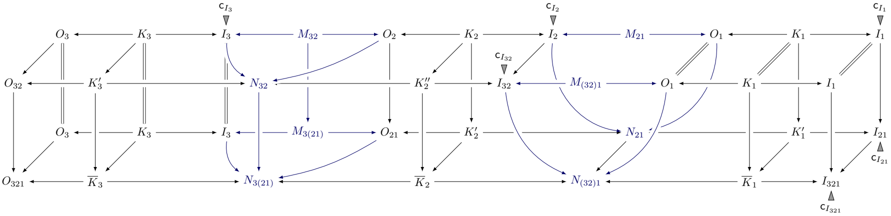

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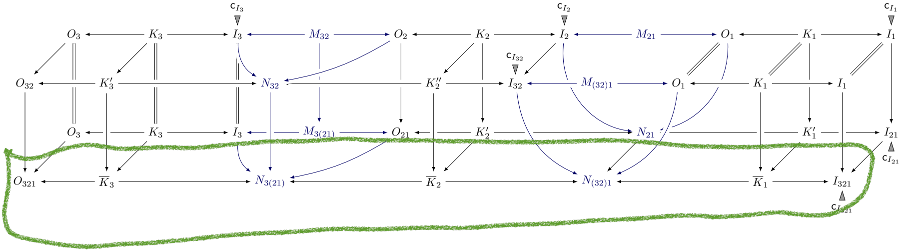

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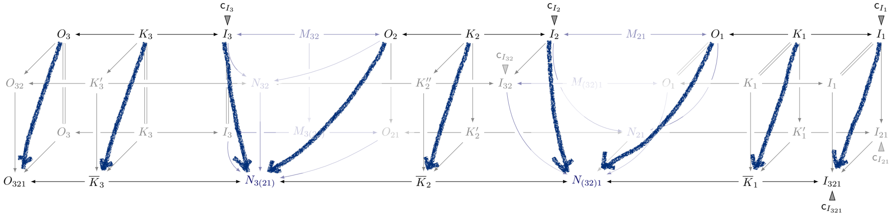

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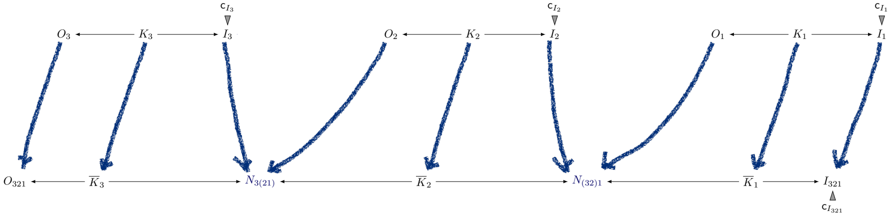
Key property: compositional associativity [6], [7], [8]



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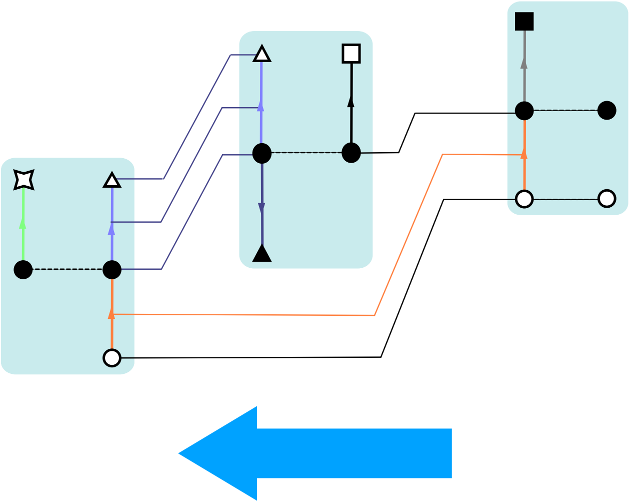

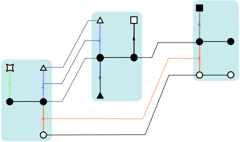

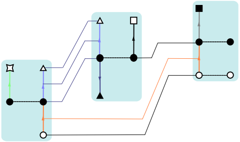

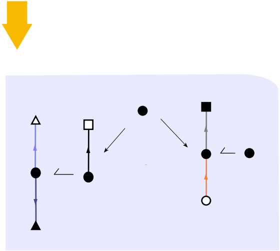

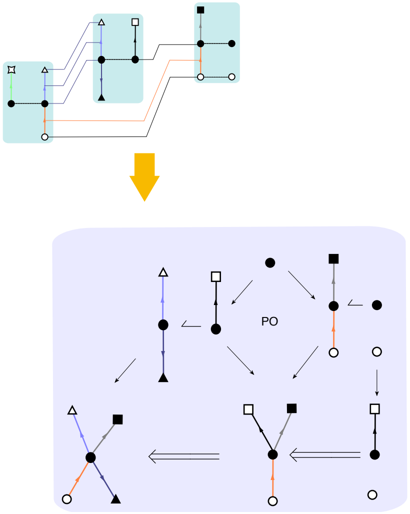
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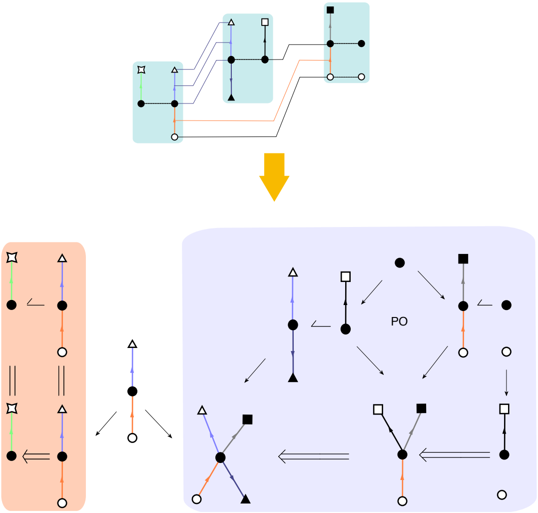
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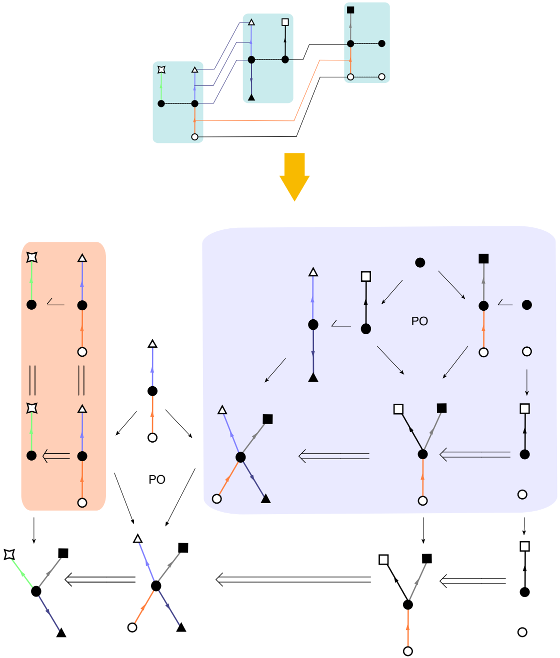
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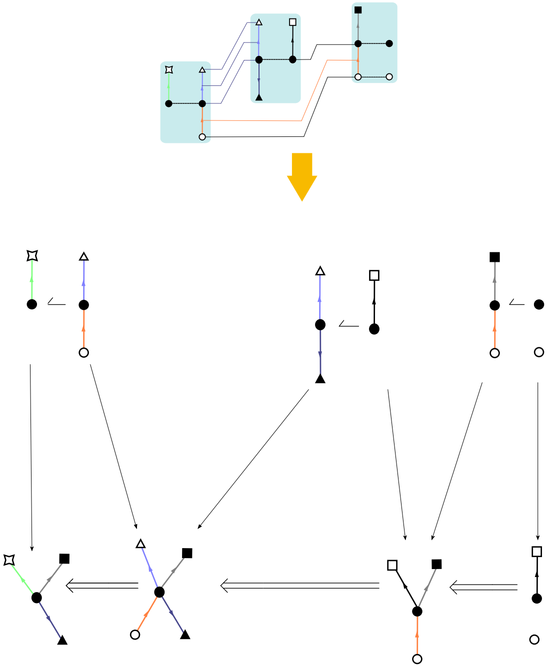
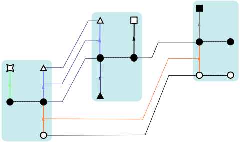
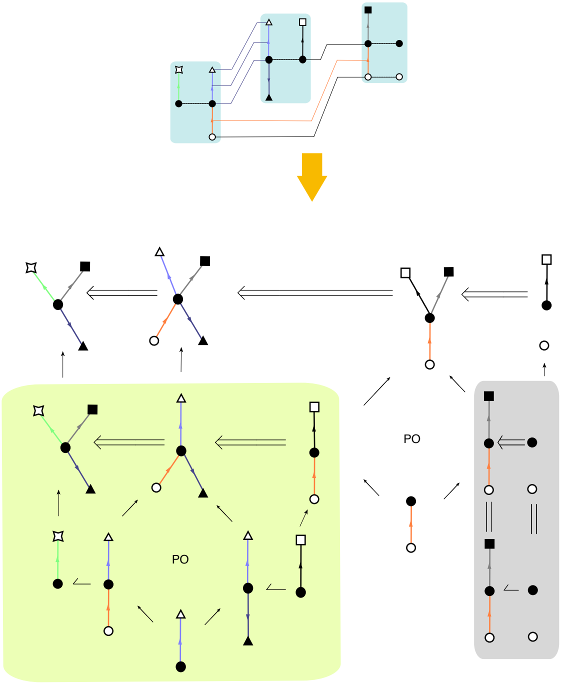
PO



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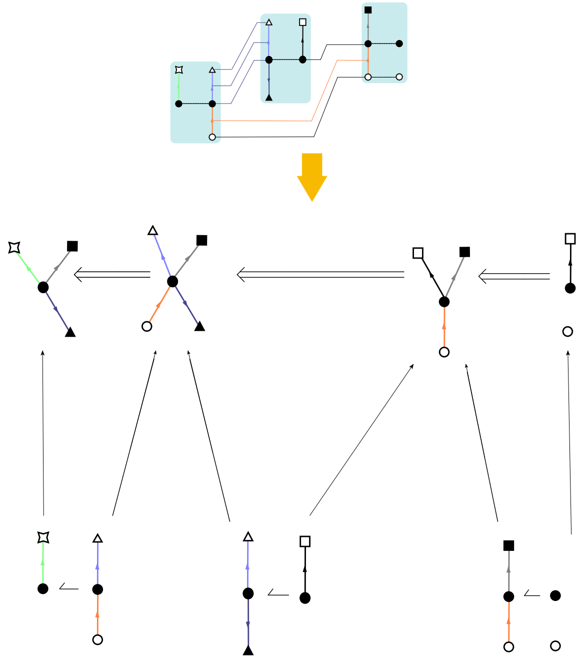
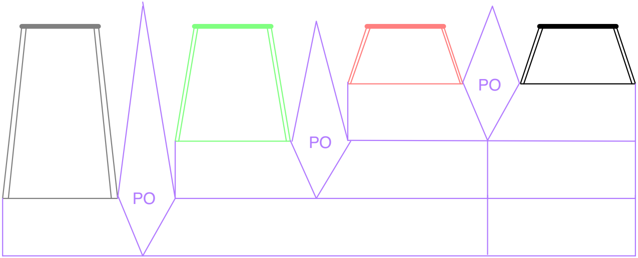
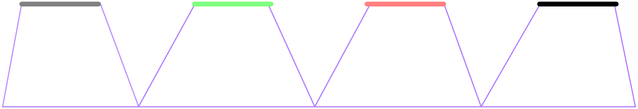


Tracelet generation

1



2



Tracelets – "generative" definition [9]

Let $\mathbb{T} \in \{DPO, SqPO\}$ be the type of rewriting, and let $\overline{\mathbf{Lin}}(\mathbf{C})$ denote the set of linear rules with conditions over $\mathbf{C}$.

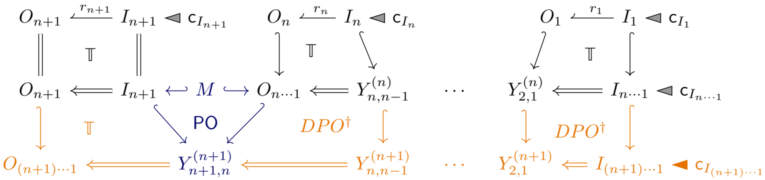
Tracelets of length 1: the set $\mathscr{T}_1^{\mathbb{T}}$ of type $\mathbb{T}$ tracelets $T(R)$ of length 1 is defined as

$$\mathscr{T}_1^{\mathbb{T}} := \left\{ T(R) = \begin{array}{ccccc} O & \xleftarrow{r} & I & \lhd & \mathsf{c}_I \\ \| & \mathbb{T} & \| & & \\ O & \Longleftarrow & I & \lhd & \mathsf{c}_I \end{array} \middle| R = (r, \mathbf{c}_I) \in \overline{\mathbf{Lin}}(\mathbf{C}) \right\}.$$

[9] Nicolas Behr. "Tracelets and Tracelet Analysis Of Compositional Rewriting Systemss". In: *arXiv preprint arXiv:1904.12829* (2019)

Tracelets – "generative" definition [9]

Tracelets of length $n+1$: given tracelets $T_{n+1} \in \mathscr{T}_1^{\mathbb{T}}$ of length 1 and $T_{n\cdots 1} \in \mathscr{T}_n^{\mathbb{T}}$ of length n (for $n \geqslant 1$), we define a span of monomorphisms $\mu = (I_{n+1} \hookleftarrow M \hookrightarrow O_{n\cdots 1})$ as $\mathbb{T}$**-admissible**, denoted $\mu \in \mathbf{MT}_{T_1}^{\mathbb{T}}(T_{n\cdots 1})$, if the following diagram is constructible:



Constructibility **may fail** due to non-existence of the requisite pushout complements, or because the tentative composite condition $\mathbf{c}_{I_{(n+1)\cdots 1}}$ might evaluate to **false**, with

$$\begin{aligned}\mathbf{c}_{I_{(n+1)\cdots}} := \mathbf{Shift}(I_{n\cdots 1} \hookrightarrow I_{(n+1)\cdots 1}, \mathbf{c}_{I_{n\cdots 1}}) \\ \bigwedge \mathbf{Trans}(Y_{n+1,n}^{(n+1)} \Leftarrow I_{(n+1)\cdots 1}, \mathbf{Shift}(I_{n+1} \hookrightarrow Y_{n+1,n}^{(n+1)}, \mathbf{c}_{I_{n+1}}))\,.\end{aligned}$$

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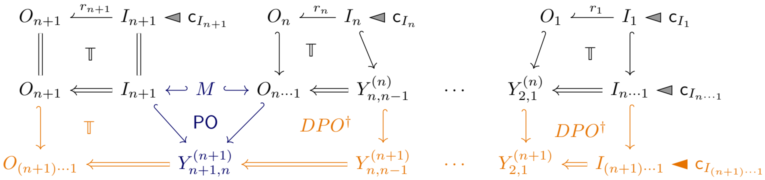
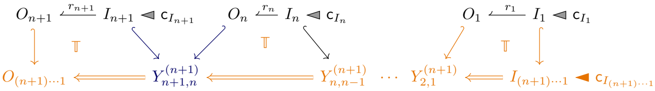


If $\mu \in \mathbf{MT}_{T_1}^{\mathbb{T}}(T_{n\cdots 1})$, we define a tracelet $T_{n+1}{}^{\mu}\angle_{\mathbb{T}} T_{n\cdots 1}$ of length $n+1$ as

$$T_{n+1}{}^{\mu}\angle_{\mathbb{T}} T_{n\cdots 1} :=$$



We define the set $\mathscr{T}_{n+1}^{\mathbb{T}}$ of **type $\mathbb{T}$ tracelets of length $n+1$** as

$$\mathscr{T}_{n+1}^{\mathbb{T}} := \left\{ T_{n+1}{}^{\mu}\angle_{\mathbb{T}} T_{n\cdots 1} \,\middle|\, T_{n+1} \in \mathscr{T}_1^{\mathbb{T}},\, T_{n\cdots 1} \in \mathscr{T}_n^{\mathbb{T}},\, \mu \in \mathbf{MT}_{T_1}^{\mathbb{T}}(T_{n\cdots 1}) \right\}.$$

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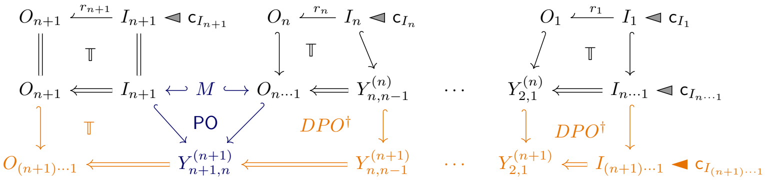

For later convenience, we introduce the **tracelet evaluation operation** [[.]],

$$[[.]] : \mathscr{T}^{\mathbb{T}} \to \overline{\mathbf{Lin}}(\mathbf{C}) : \mathscr{T}_n^{\mathbb{T}} \ni T \mapsto [[T]] := \left((O_{n\cdots1} \leftharpoonup I_{n\cdots1}), \mathbf{c}_{I_{n\cdots1}}\right),$$

with $\mathscr{T}^{\mathbb{T}} := \bigcup_{n\geqslant1} \mathscr{T}_n^{\mathbb{T}}$, and where $(O_{n\cdots1} \leftharpoonup I_{n\cdots1})$ denotes the span composition

$$(O_{n\cdots1} \leftharpoonup I_{n\cdots1}) := (O_{n\cdots1} \Leftarrow Y^{(n)}_{n,n-1}) \circ \cdots \circ (Y^{(n)}_{2,1} \Leftarrow I_{n\cdots1}).$$

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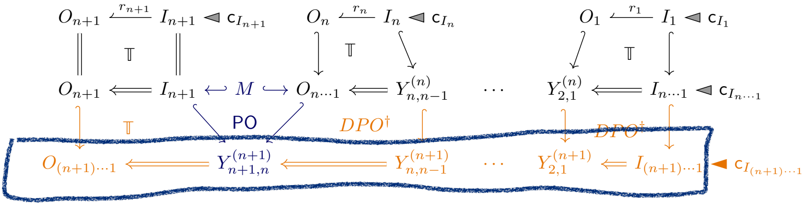

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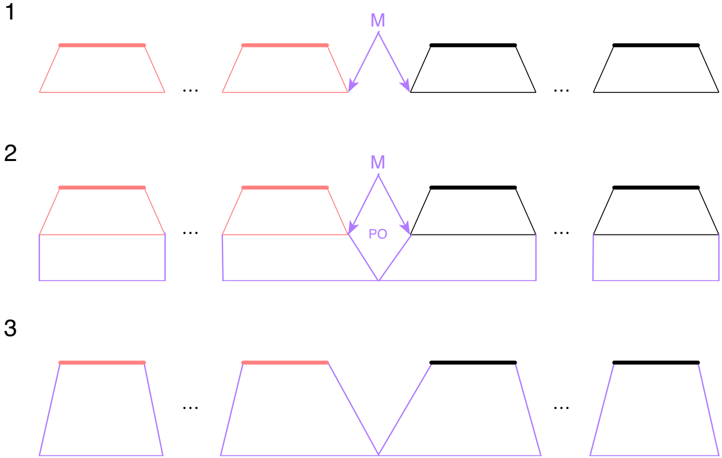
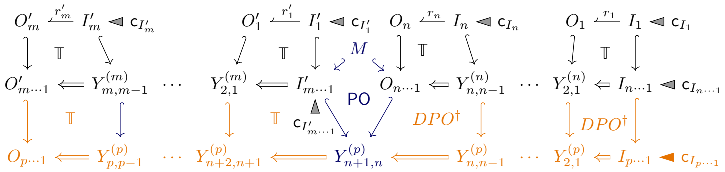
Tracelet composition



Tracelet composition [9]

For tracelets $T', T \in \mathcal{T}^{\mathbb{T}}$ of lengths m and n, respectively, a span of monomorphisms $\mu = (I'_{m\cdots 1} \hookleftarrow M \hookrightarrow O_{n\cdots 1})$ is defined to be an **admissible match of T into T'**, denoted $\mu \in \mathbf{MT}^{\mathbb{T}}_{T'}(T)$, if (i) all requisite pushout complements exist to form the type $DPO^{\dagger}$ derivations (in the sense of rules without conditions) to construct the diagram below, where $p := m + n + 1$,



and if (ii) the condition $\mathbf{c}_{I_{(m+n+1)\cdots 1}}$ below does not evaluate to **false**:

$$\mathbf{c}_{I_{(m+n+1)\cdots}} := \mathbf{Shift}(I_{n\cdots 1} \hookrightarrow I_{(m+n+1)\cdots 1}, \mathbf{c}_{I_{n\cdots 1}}) \bigwedge \mathbf{Trans}(Y^{(m+n+1)}_{n+1,n} \Leftarrow I_{(m+n+1)\cdots 1}, \mathbf{Shift}(I_{m\cdots 1} \hookrightarrow Y^{(n+1)}_{n+1,n}, \mathbf{c}_{I_{m\cdots 1}}))\,.$$

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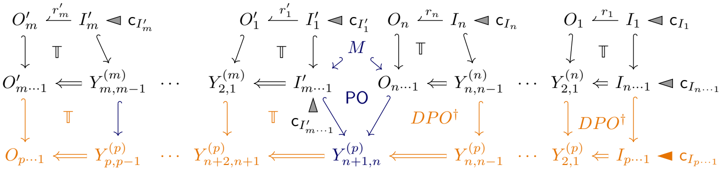
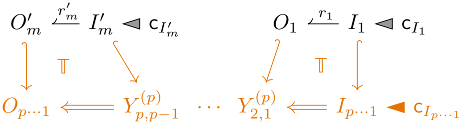


Then for $\mu \in \mathbf{MT}^{\mathbb{T}}_{T'}(T)$, we define the **type $\mathbb{T}$ tracelet composition of T' with T along μ** as

$T'^{\mu}\angle_{\mathbb{T}} T :=$



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Theorem: properties of the tracelet composition operation [9]

Let $.\triangleleft_{\mathbb{T}}.$ denote the $\mathbb{T}$-type rule composition, and let the set of $\mathbb{T}$-admissible matches be denoted by $\mathbf{M}^{\mathbb{T}}_{r_2}(r_1)$ (for $r_2, r_1 \in \overline{\mathbf{Lin}}(\mathbf{C})$).

(i) For all $T', T \in \mathscr{T}^{\mathbb{T}}$, $\mathbf{MT}^{\mathbb{T}}_{T'}(T) = \mathbf{M}^{\mathbb{T}}_{[[T']]}([[T]])$.

(ii) For all $T', T \in \mathscr{T}^{\mathbb{T}}$ and $\mu \in \mathbf{MT}^{\mathbb{T}}_{T'}(T)$, $\left[\left[T'^{\mu}\angle_{\mathbb{T}} T\right]\right] = [[T']]^{\mu}\triangleleft_{\mathbb{T}}[[T]]$.

(iii) The $\mathbb{T}$-type tracelet composition is **associative**, i.e. for any three tracelets $T_1, T_2, T_3 \in \mathscr{T}^{\mathbb{T}}$, there exists a bijection $\varphi : S_{3(21)} \xrightarrow{\cong} S_{(32)1}$ between the sets pairs of $\mathbb{T}$-admissible matches of tracelets (with $T_{ji} := T_j{}^{\mu_{ji}}\angle_{\mathbb{T}} T_i$ and using property (i))

$$S_{3(21)} := \{(\mu_{21}, \mu_{3(21)}) | \mu_{21} \in \mathbf{M}^{\mathbb{T}}_{[[T_2]]}([[T_1]]),\ \mu_{3(21)} \in \mathbf{M}^{\mathbb{T}}_{[[T_3]]}([[T_{21}]])$$
$$S_{(32)1} := \{(\mu_{32}, \mu_{(32)1}) | \mu_{32} \in \mathbf{M}^{\mathbb{T}}_{[[T_3]]}([[T_2]]),\ \mu_{(32)1} \in \mathbf{M}^{\mathbb{T}}_{[[T_{32}]]}([[T_1]])\}$$

such that for all $(\mu'_{32}, \mu'_{(32)1}) = \varphi((\mu_{21}, \mu_{3(21)}))$

$$T_3{}^{\mu_{3(21)}}\angle_{\mathbb{T}} (T_2{}^{\mu_{21}}\angle_{\mathbb{T}} T_1) \cong \left(T_3{}^{\mu'_{32}}\angle_{\mathbb{T}} T_2\right)^{\mu'_{(32)1}}\angle_{\mathbb{T}} T_1 .$$

Moreover, the bijection φ coincides with the corresponding bijection provided in the associativity theorem for $\mathbb{T}$-type rule compositions.

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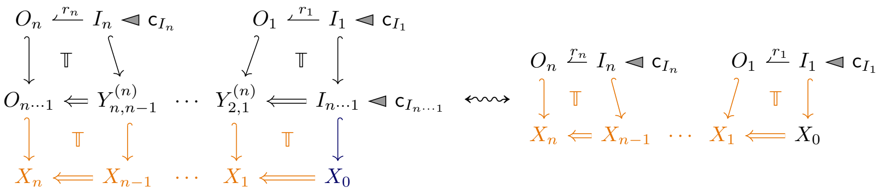
Tracelet characterization theorem [9]

For all type-$\mathbb{T}$ tracelets $T \in \mathscr{T}_n^{\mathbb{T}}$ of length n, for all objects X_0 of $\mathbf{C}$, and for all monomorphisms $(m : I_{n\cdots 1} \hookrightarrow X_0)$ such that $m \in \mathbf{M}^{\mathbb{T}}_{[[T]]}(X_0)$, there exists a type-$\mathbb{T}$ direct derivation $D = T_m(X_0)$ obtained via vertically composing the squares in each column of the diagram below:



Conversely, every $\mathbb{T}$-direct derivation D of length n along rules $R_j = (r_j, \mathbf{c}_{I_j}) \in \overline{\mathbf{Lin}}(\mathbf{C})$ starting at an object X_0 of $\mathbf{C}$ may be cast into the form $D = T_m(X_0)$ for some tracelet T of length n and a $\mathbb{T}$-admissible match $m \in \mathbf{M}^{\mathbb{T}}_{[[T]]}(X_0)$ that are uniquely determined from D (up to isomorphisms).

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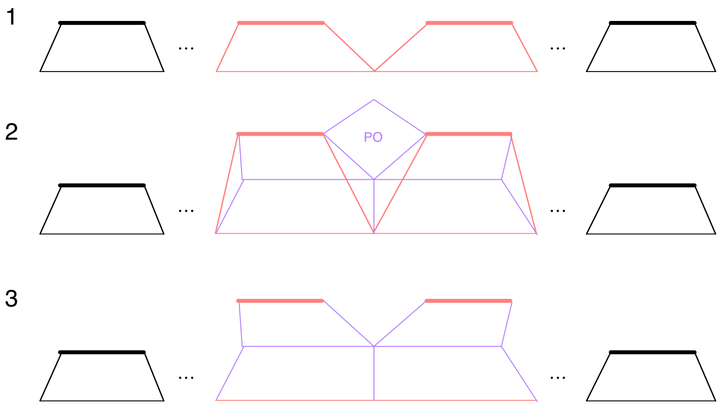
Tracelet analysis



Convenient shorthand notation: subtracelets

$$T \equiv t_n|\ldots|t_1 = \begin{array}{ccccccc} O_n & \overset{r_n}{\hookleftarrow} & I_n \blacktriangleleft \mathsf{c}_{I_n} & & O_1 & \overset{r_1}{\hookleftarrow} & I_1 \blacktriangleleft \mathsf{c}_{I_1} \\ \downarrow & \mathbb{T} & \searrow & & \swarrow & \mathbb{T} & \downarrow \\ O_{n\cdots 1} & \Longleftarrow & Y^{(n)}_{n,n-1} & \cdots & Y^{(n)}_{2,1} & \Longleftarrow & I_{n\cdots 1} \blacktriangleleft \mathsf{c}_{I_{n\cdots 1}} \end{array}$$

For a tracelet $T \in \mathscr{T}_n^{\mathbb{T}}$ of length $n \geqslant 1$, let symbols t_j for $1 \leqslant j \leqslant n$ denote **j-th subtracelets** of T, so that $T \equiv t_n|t_{n-1}|\ldots|t_1$ is a concatenation of its subtracelets, with

$$t_j := \begin{array}{ccc} O_j & \overset{r_j}{\hookleftarrow} & I_j \blacktriangleleft \mathsf{c}_{I_j} \\ \downarrow & \mathbb{T} & \downarrow \\ Y^{(n)}_{j+1,j} & \Longleftarrow & Y^{(n)}_{j,j-1} \end{array} \quad , \qquad Y^{(n)}_{n+1,n} := O_{n\cdots 1}\,,\; Y^{(n)}_{1,0} := I_{n\cdots 1}\,.$$

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t_1

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\downarrow & \mathbb{T} & \searrow & & \swarrow & & \mathbb{T} & \downarrow \\
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\end{array}
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tn

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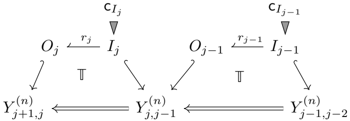
$$

Corollary: tracelet surgery

Let $T \in \mathscr{T}_n^{\mathbb{T}}$ a $\mathbb{T}$-type tracelet of length n, so that $T \equiv t_n|\ldots|t_1$. Then for any consecutive subtracelets $t_j|t_{j-1}$ in T, one may uniquely (up to isomorphisms) construct a diagram $t_{(j|j-1)}$ and a tracelet $T_{(j|j-1)}$ of length 2 as follows:



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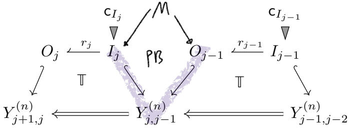

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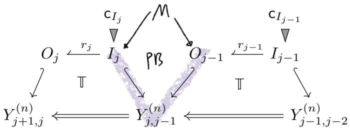

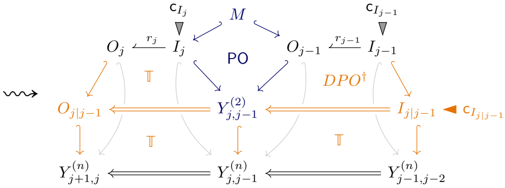

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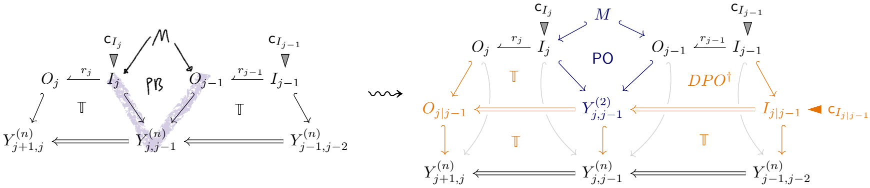

$$t_{(j|j-1)} := \begin{array}{ccc} O_{j|j-1} & \longleftarrow & I_{j|j-1} \blacktriangleleft \mathsf{c}_{I_{j|j-1}} \\ \downarrow & \mathbb{T} & \downarrow \\ Y^{(n)}_{j+1,j} & \Longleftarrow & Y^{(n)}_{j-1,j-2} \end{array} \quad , \quad T_{(j|j-1)} := T(r_j, \mathbf{c}_{I_j})^{\mu} \angle_{\mathbb{T}} T(r_{j-1}, \mathbf{c}_{I_{j-1}})$$

Here, $\mu = (I_j \hookleftarrow M \hookrightarrow O_{j-1})$ is the span of monomorphisms obtained by taking the pullback of the cospan $(I_j \hookrightarrow Y^{(n)}_{j,j-1} \hookleftarrow O_{j-1})$, and this μ is always a $\mathbb{T}$-admissible match. By associativity of the tracelet composition, this extends to consecutive sequences $t_j|\dots|t_{j-k}$ of subtracelets in T inducing diagrams $t_{(j|\dots|j-k)}$ and tracelets of length 1 $T_{(j|\dots|j-k)}$, where for $k = 0$, $t_{(j)} = t_j$ and $T_{(j)} = T(r_j, \mathbf{c}_{I_j})$.

Tracelet abstraction equivalence

Two tracelets $T, T' \in \mathscr{T}_n^{\mathbb{T}}$ of the same length $n \geqslant 1$ are defined to be **abstraction equivalent**, denoted $T \equiv_A T'$, if there exist suitable isomorphisms on the objects in T in order to transform T into T' (with transformations on morphisms induced by object isomorphisms).

Tracelet shift equivalence

Let $T, T' \in \mathscr{T}_n^{\mathbb{T}}$ be two tracelets of the same length $n \geqslant 1$. If there exist subtracelets $t_j|\ldots|t_{j-k}$ and $t'_j|\ldots|t'_{j-k}$ such that

(i) the subtracelets have the same rule content (up to isomorphisms), i.e. there exists a permutation $\sigma \in S_k$ such that $[[T_{(p)}]] \cong [[T'_{(\sigma(p)}]]$ for all $j-k \leqslant p \leqslant j$, and

(ii) the diagrams $t_1|\ldots|t_{(j|\ldots|j-k)}|\ldots|t_n$ and $t'_1|\ldots|t'_{(j|\ldots|j-k)}|\ldots|t'_n$ are isomorphic,

then T and T' are defined to be **shift equivalent**, denoted $T \equiv_S T'$. Extending $\equiv_S$ by transitivity then yields an equivalence relation on $\mathscr{T}_n^{\mathbb{T}}$ for every $n \geqslant 1$.

An arena for static analysis: "pathways" in rewriting systems

- Let $\mathscr{R} = \{R_j \in \overline{\mathbf{Lin}}(\mathbf{C})\}_{j \in J}$ a (finite) set of rules with conditions over $\mathbf{C}$, which model the **transitions** of a rewriting system.
- We designate a rule $E \in \overline{\mathbf{Lin}}(\mathbf{C})$ as modeling a **"target event"**, i.e. E must be the last rule applied in the derivation traces we will study.
- Let moreover $\equiv_C$ be an equivalence relation on derivation traces such as abstraction or shift equivalences, or combinations thereof.

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"Pathway generation" or "explanatory synthesis" problem

For the type-$\mathbb{T}$ rewriting system based upon the set of rules $\mathscr{R}$, synthesize the **maximally compressed** derivation traces ending in an application of E such that "E cannot occur at an earlier position in a given trace". Here, compression refers to retaining only the smallest traces in a given $\equiv_C$ equivalence class, while the last part of the statement needs to be made precise in a specific application (as it depends on the chosen framework).

Feature-driven Explanatory Traclet Analysis (FETA)

- $\equiv_C$ — conjunction of **tracelet abstraction and shift equivalences** $\equiv_A$ and $\equiv_S$
- For $T = t_E|t_n|\ldots|t_1 \in \mathscr{T}^{\mathbb{T}}_{n+1}$ (with t_E containing the rule E, $[[T_{(E)}]] \cong E$), let $E \prec_C T$ denote the following property: there exist no tracelets $T' \in \mathscr{T}^{\mathbb{T}}_{n+1}$

$$t_E|t_n|\ldots|t_1 \equiv_C t'_{n+1}|t'_n|\ldots|t'_1 \quad \text{with} \quad [[T'_{(k)}]] \cong E \text{ for an index } k < n+1\,.$$

$\Rightarrow$ set of **strongly compressed pathways** := set of such tracelets modulo $\equiv_C$

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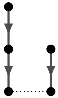
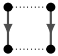
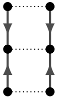
Algorithm 1: Feature-driven Explanatory Tracelet Analysis (FETA)

Data: $N_{max} \geq 2 \leftarrow$ maximal length of tracelets to be generated
$T_E := T(E) \leftarrow$ tracelet of length 1 associated to the rule E
$\mathsf{T}_1 := \{T(R_j) \mid j \in J\} \leftarrow$ set of tracelets of length 1 associated to the transitions
Result: sets P_i $(i = 2, \dots, N_{max})$ of strongly compressed pathways
begin
    $P_1 := \{T_E\} \leftarrow$ the only pathway of length 1;
    for $2 < n \leq N_{max}$ **do**
        $\mathsf{pre}_n := \left\{P {}^{\mu}\!\angle_{\mathbb{T}} T \mid P \in \mathsf{P}_{n-1}, T \in \mathsf{T}_1\,,\ \mu \in \mathsf{MT}^{\mathbb{T}}_P(T)\right\}$;
        $\mathsf{P}_n := \{T' \in \mathsf{pre}_n | E \prec_C T'\}\diagup_{\equiv_C}$;
    end
end

Prototypical example: a rewriting system in FinGraph

Let $\mathbf{C} = \mathbf{FinGraph}$ be the category of finite directed multigraphs. Let $\mathscr{R} = \{r\}$ be a one-element **transition set** (for a rule $r \in \mathbf{Lin}(\mathbf{FinGraph})$ without conditions), and let $e_1, e_2 \in \mathbf{Lin}(\mathbf{FinGraph})$ be two rules modeling alternative **target events**:

$r =$  , $e_1 =$  , $e_2 =$  .

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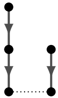
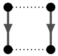
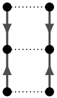
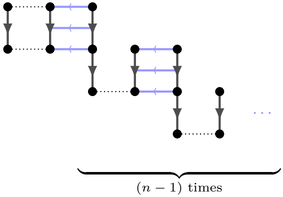
$r =$  , $e_1 =$  , $e_2 =$  .

If we consider **DPO-type** rewriting, the FETA algorithm produces the following strongly compressed pathways for **target event** e_1 and $n \geqslant 2$ (with light blue arrows indicating the relative overlap structure within the tracelets):

$$\mathbf{P}_n = \{S_n\}\,, \qquad S_n = t_E | \underbrace{t_r | \ldots | t_r}_{(n-1)\ \text{times}} \ =$$



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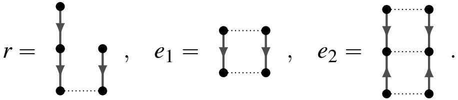

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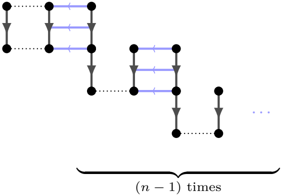
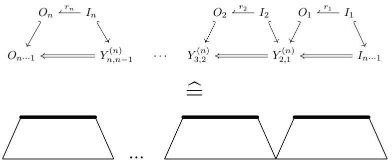
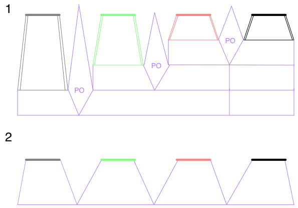


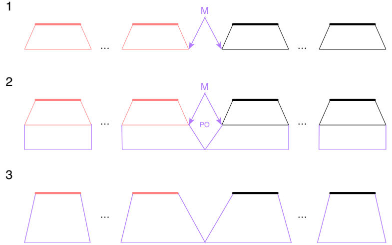
For the **target event** e_2 the algorithm detects **no pathways** $\mathbf{P}'_n$ for $n \geqslant 2$.



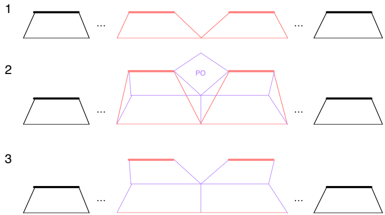
(a) Tracelets as (minimal) derivation traces.



(b) Tracelet generation (Definition 2.1).



(c) Tracelet composition (Definition 2.2).



(d) Tracelet analysis (Section 3).

Figure 2 Schematic overview of the tracelet and tracelet analysis framework.

Thank you!

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