Combinatorics of Non-Linear Rewriting Theories via Double-Categorical Methods

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Joint work with:

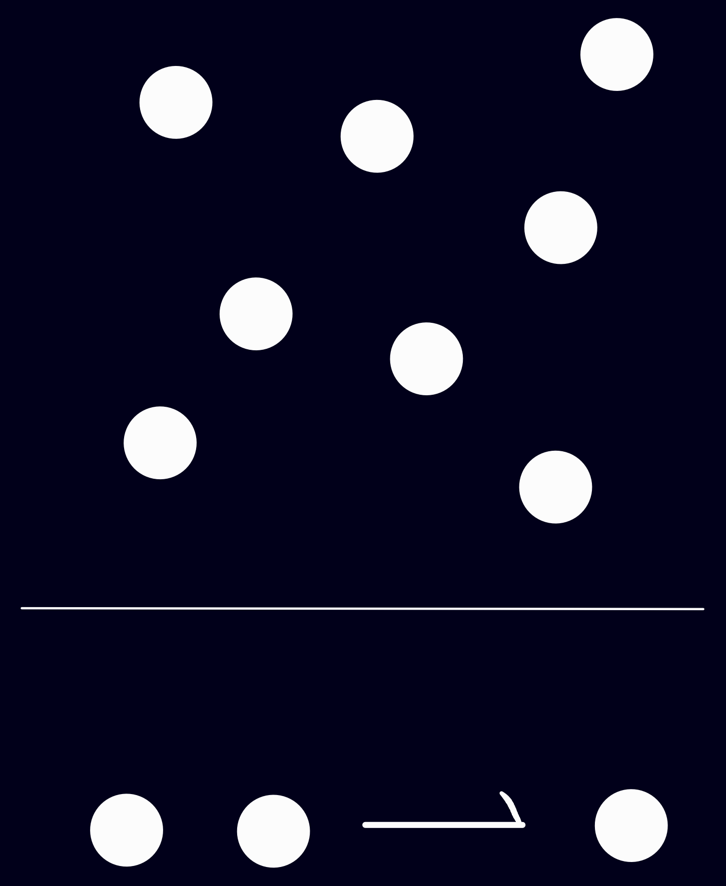
- Russ Harmer (ENS Lyon)
- Paul-André Melliès (IRIF) and Noam Zeilberger (LIX)
- Hélène Han (ENS Paris-Saclay)

↤ arXiv: 2204.07175 (under review for SCAMP)

↤ paper accepted at FSCD '23

↤ stage M1 project at IRIF

① MOTIVATION



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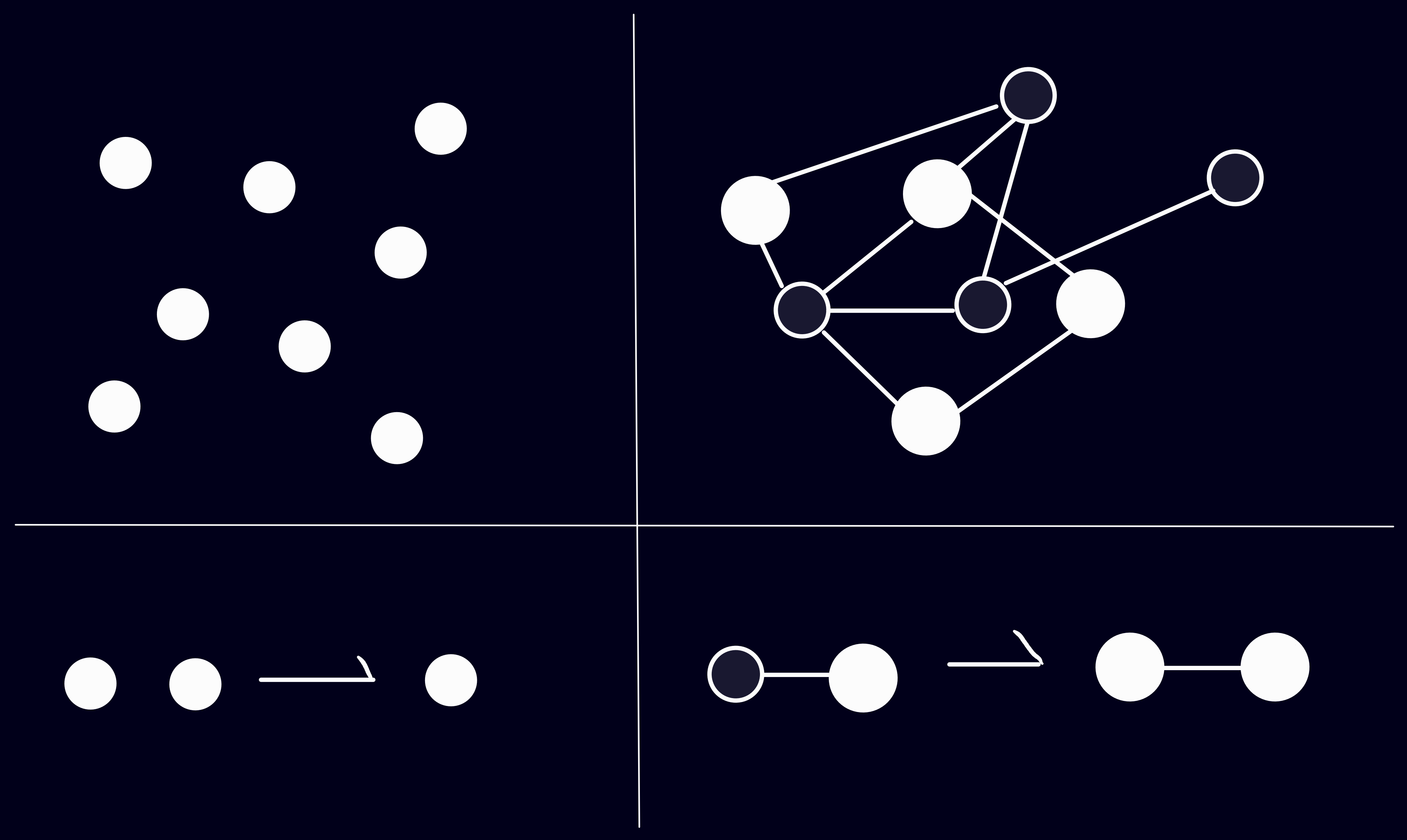

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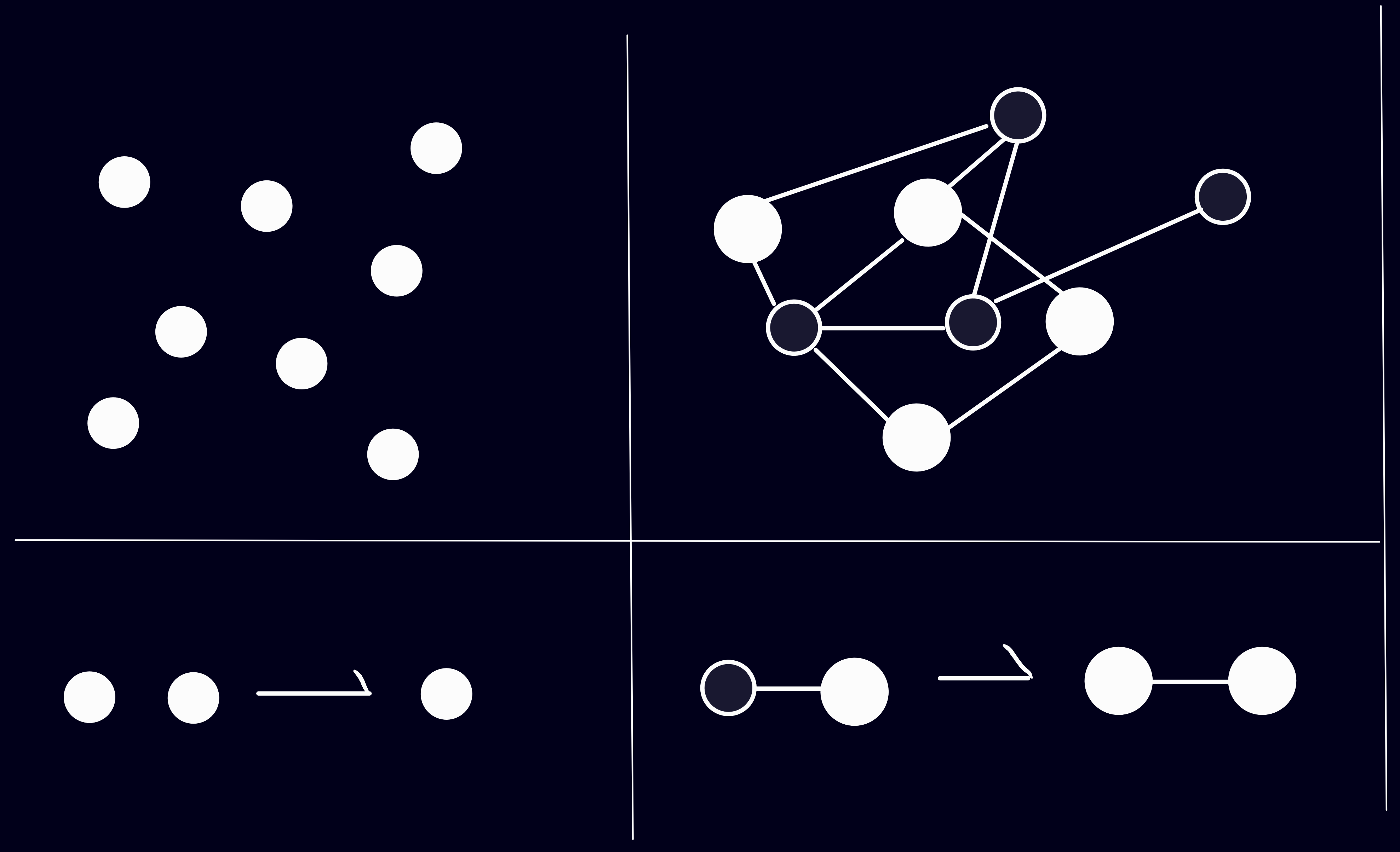
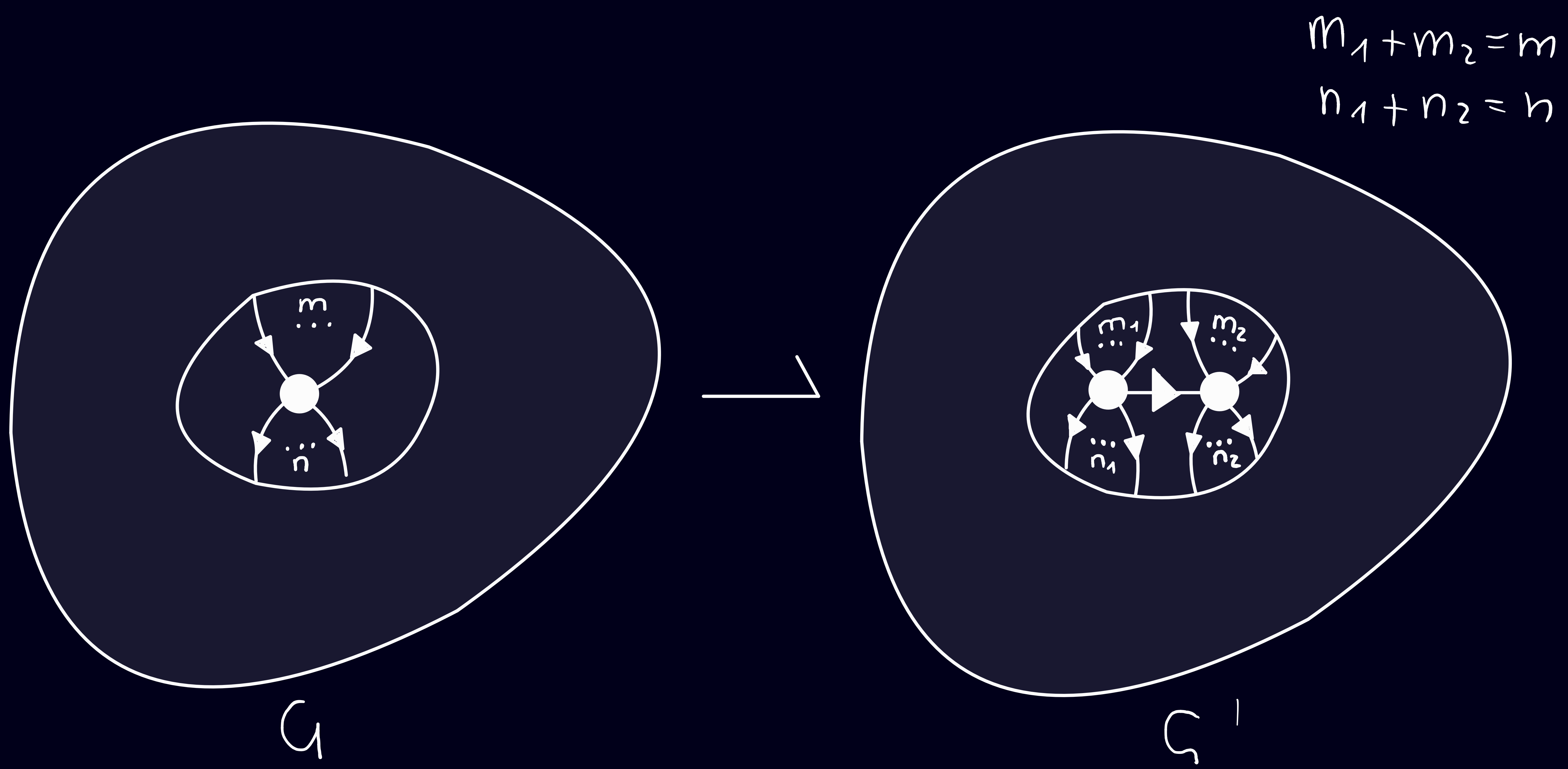
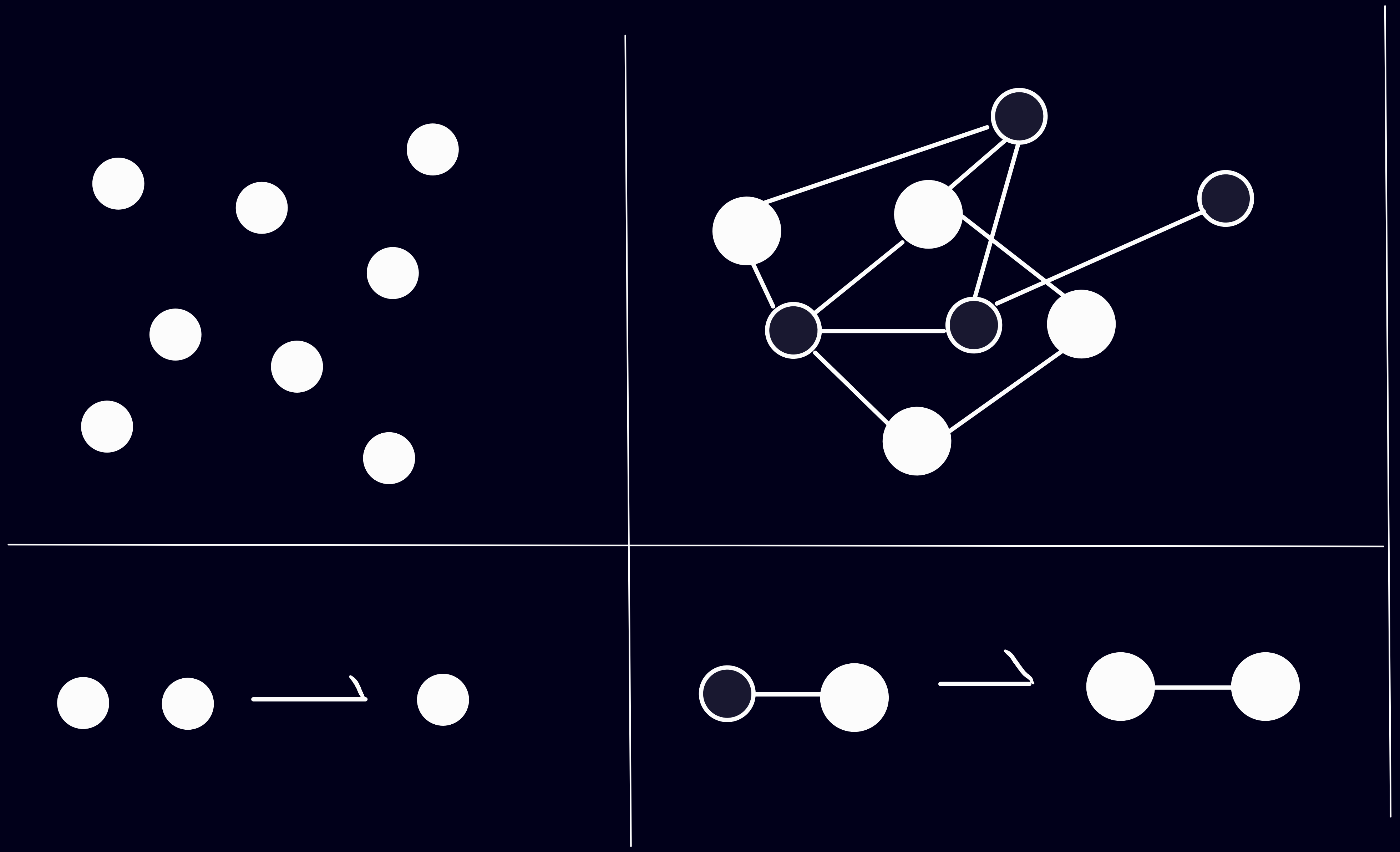
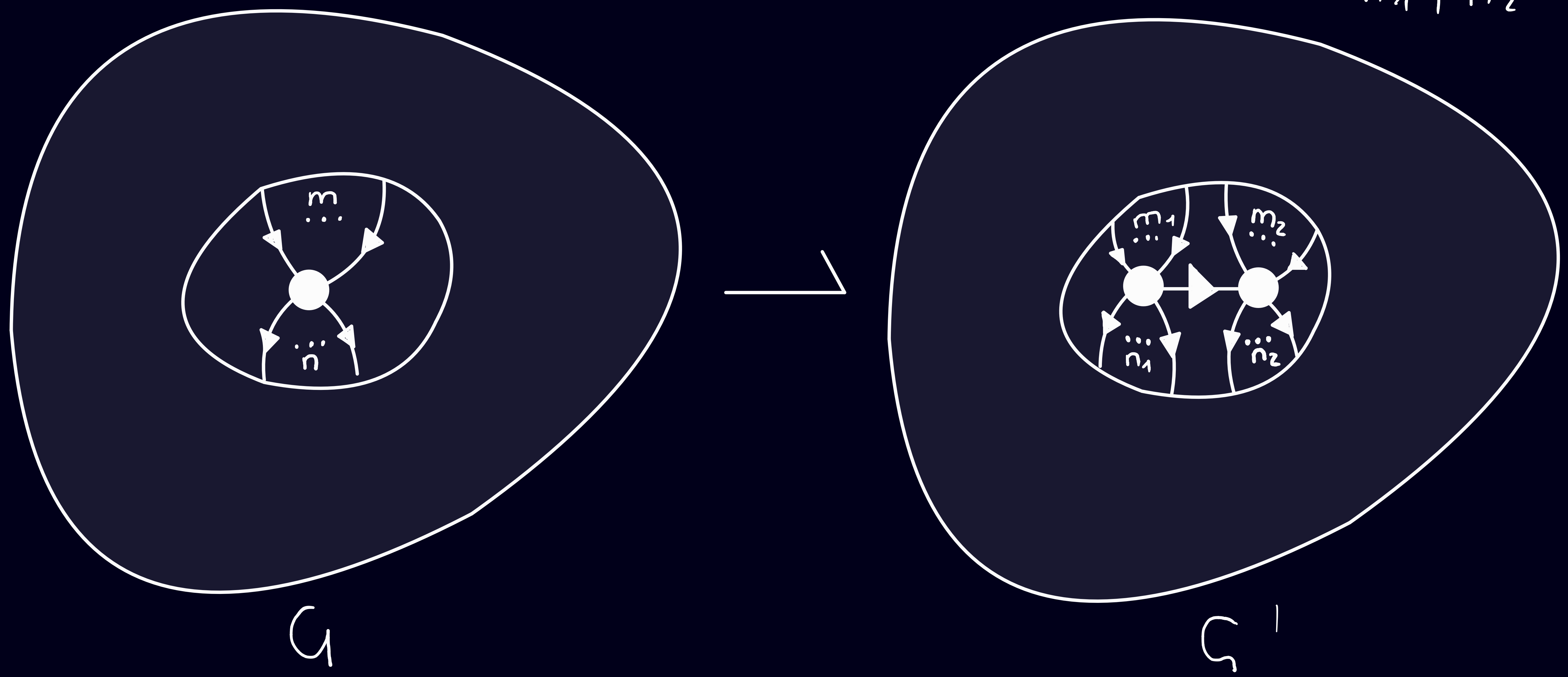




① Motivation

$m_1 + m_2 = m$
$n_1 + n_2 = n$





↪ All formalizable in Double-Pushout (DPO) semantics:

$$
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
r_\alpha(X) & X \arrow[l]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\
r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X
\end{tikzcd}
$$

PO — Pushout

② Conceptual Obstacle: Essential Uniqueness of Universal Constructions

Recap:

$$
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
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O \arrow[d, "n"'] \arrow[dr, phantom, "PO"] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dr, phantom, "PO"] & I \arrow[d, "m"] \\
r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X
\end{tikzcd}
$$

PO — Pushout

Definition:

$$
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\text{ is a PO } :\Longleftrightarrow \ \forall
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D & \\
 & & X
\end{tikzcd}
: \
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!"] & \\
 & & X
\end{tikzcd}
$$

$$
\hookrightarrow
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r] & D
\end{tikzcd}
\wedge
\begin{tikzcd}
A \arrow[r] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r] & D'
\end{tikzcd}
\Rightarrow
\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[d] \arrow[ddr, bend left] & \\
C \arrow[r] \arrow[drr, bend right] & D \arrow[dr, dashed, "\exists!", "\cong"'] & \\
 & & D'
\end{tikzcd}
$$

Example: (in Finset)

$$
\begin{tikzcd}
I \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r, hook] & \text{"}B \cup_I C\text{"}
\end{tikzcd}
$$

② Conceptual Obstacle: Essential Uniqueness of Universal Constructions

Recap:

$$
\begin{tikzcd}
O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\
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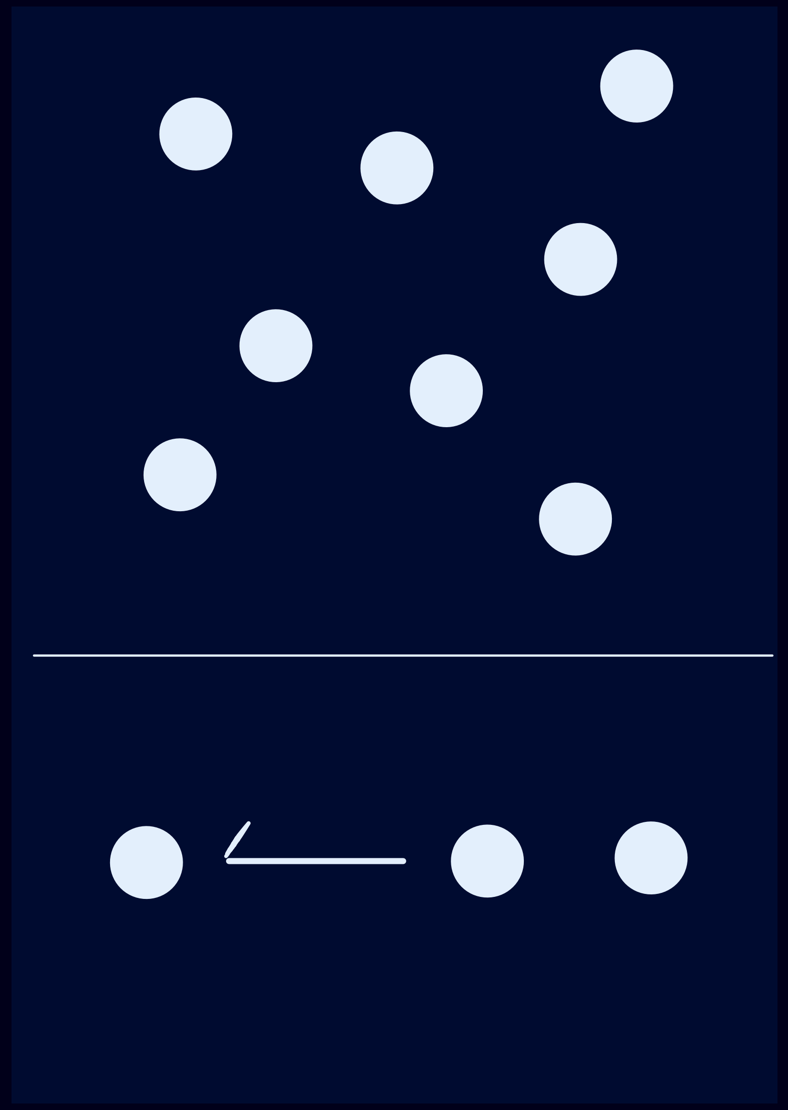
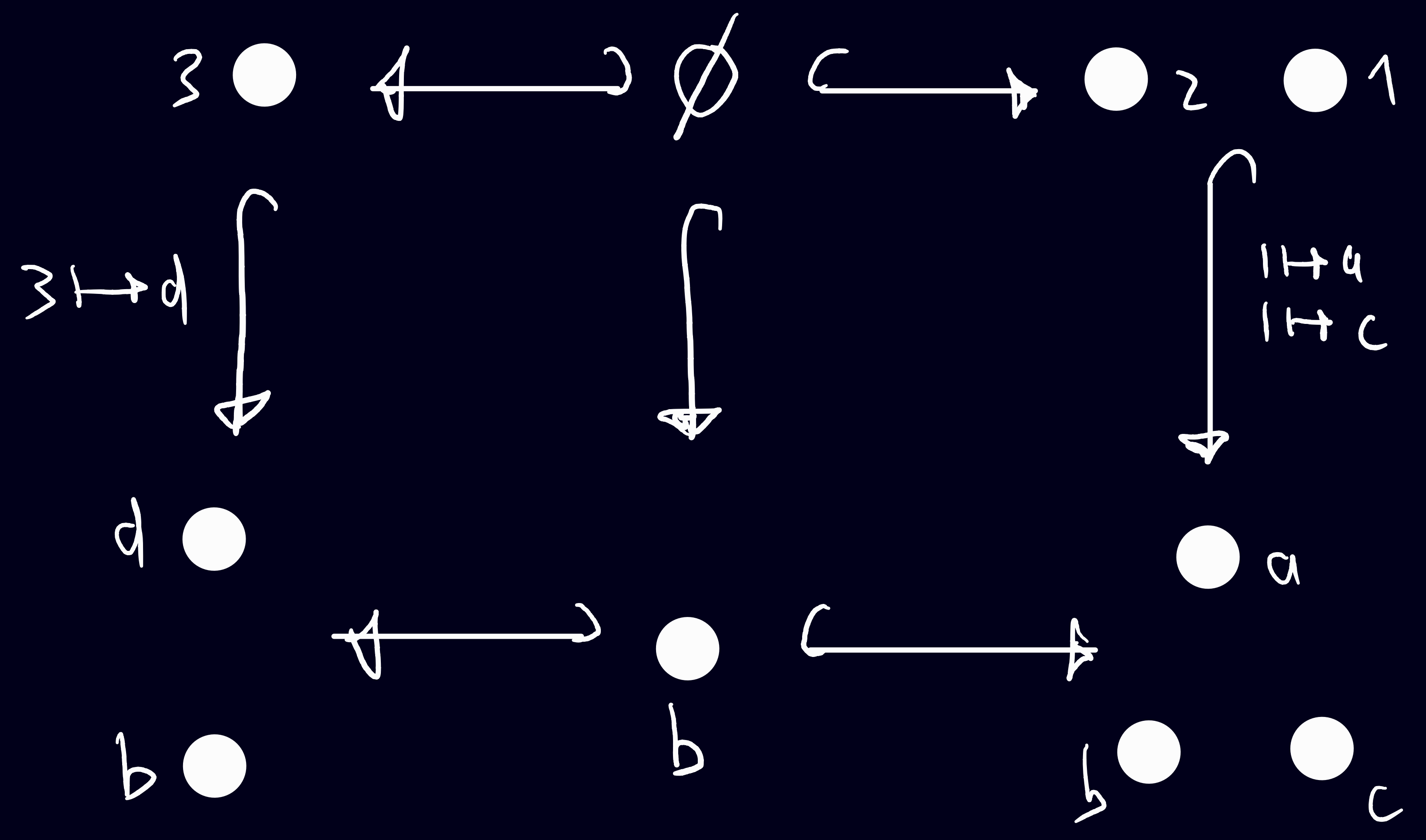
r_\alpha(X) & K_\alpha \arrow[l, "o_\alpha"] \arrow[r, "i_\alpha"'] & X
\end{tikzcd}
$$

PO — Pushout

Example:

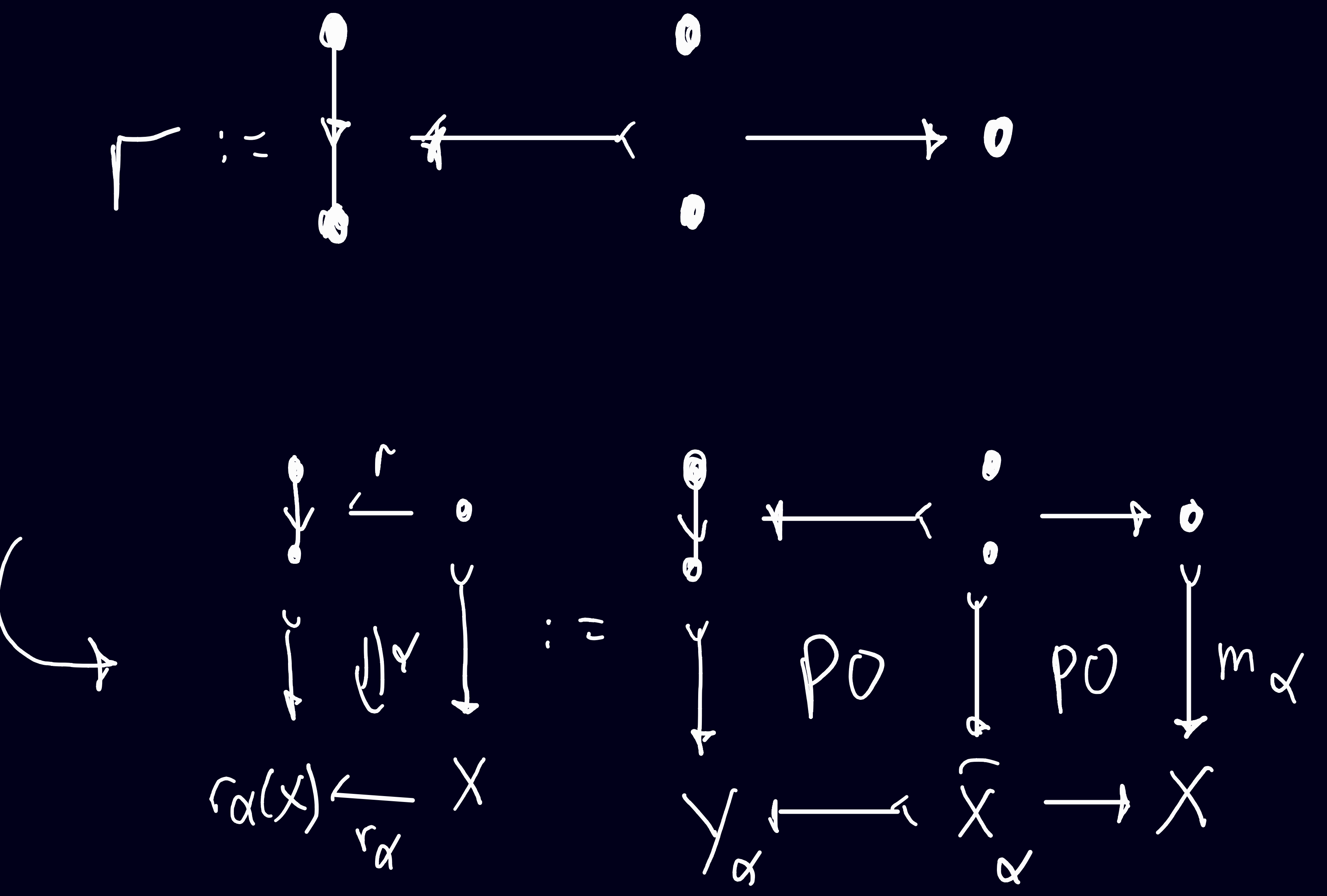


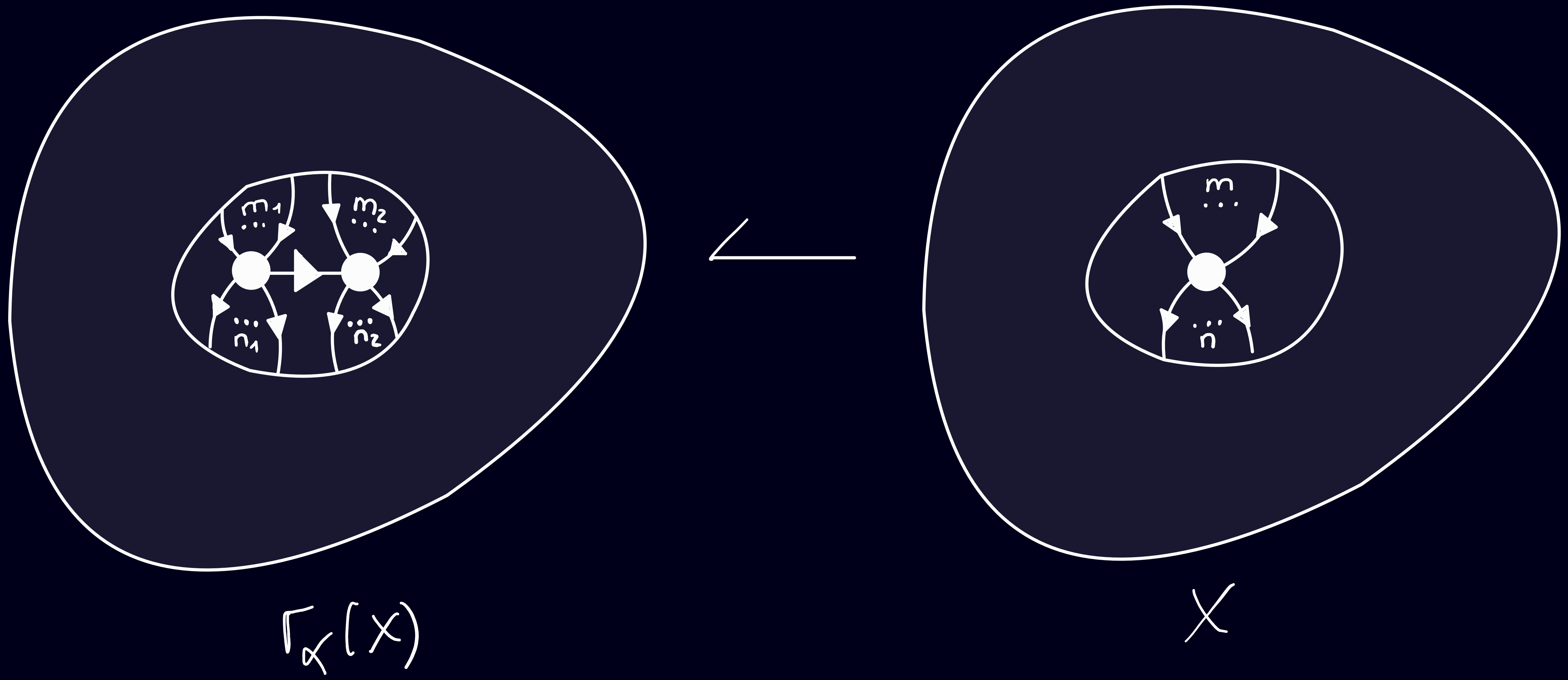


In $\underline{\text{FinSet}}$:

$$
\begin{tikzcd}
I \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "PO"] & B \arrow[d] \\
C \arrow[r, hook] & \text{"}B \cup_I C\text{"}
\end{tikzcd}
$$

③ Non-linear rules in Kontsevich's graph cohomology calculus



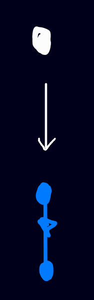


• Observation: up to isomorphisms, each rewrite is determined by:

1 Choice of vertex in X ($\bullet \xrightarrow{m_\alpha} X$)

2 Choice of a partition of edges incident to $m_\alpha(\bullet)$

③ NON-LINEAR RULES IN KONTSEVICH'S GRAPH COHOMOLOGY CALCULUS



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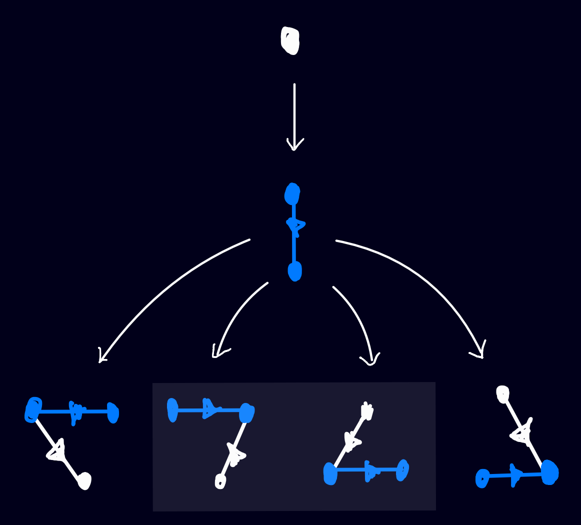

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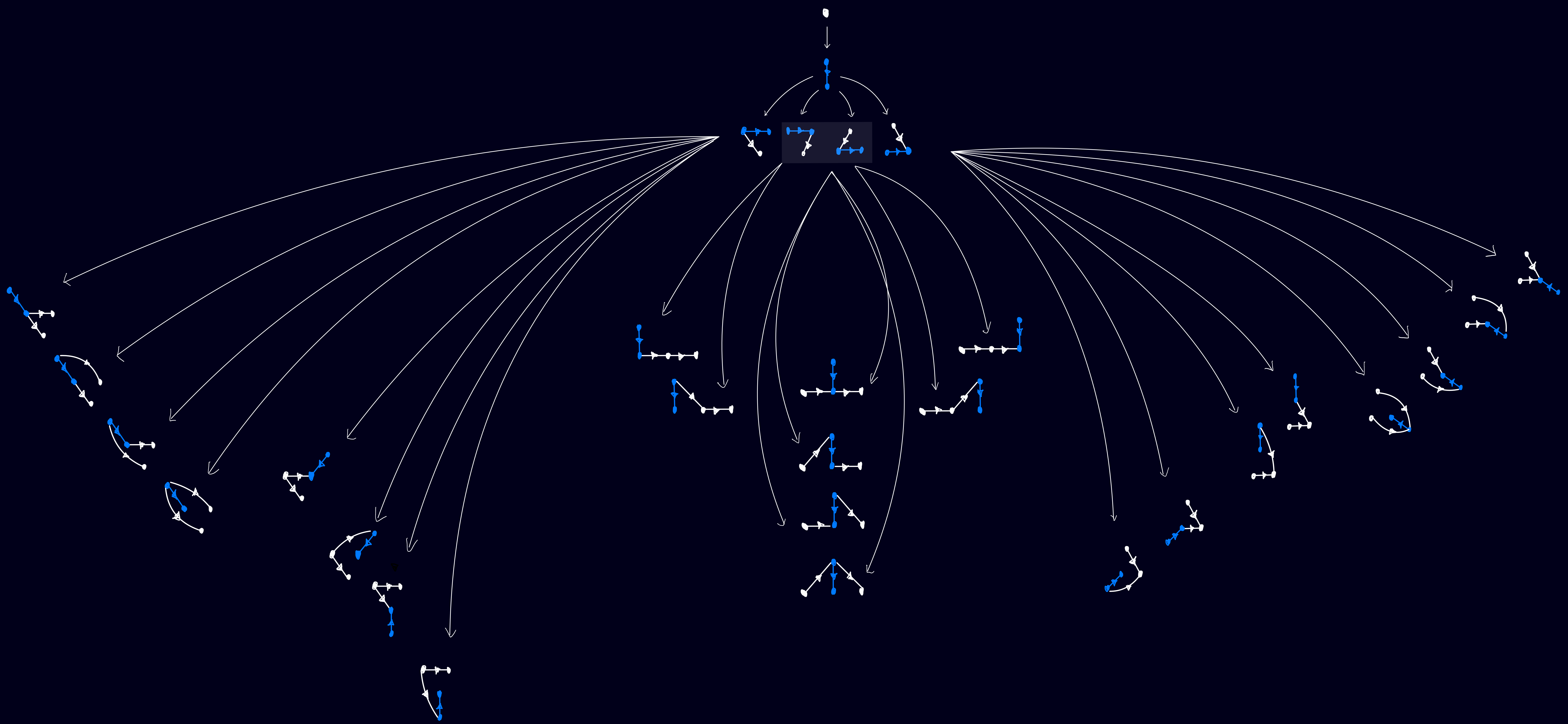

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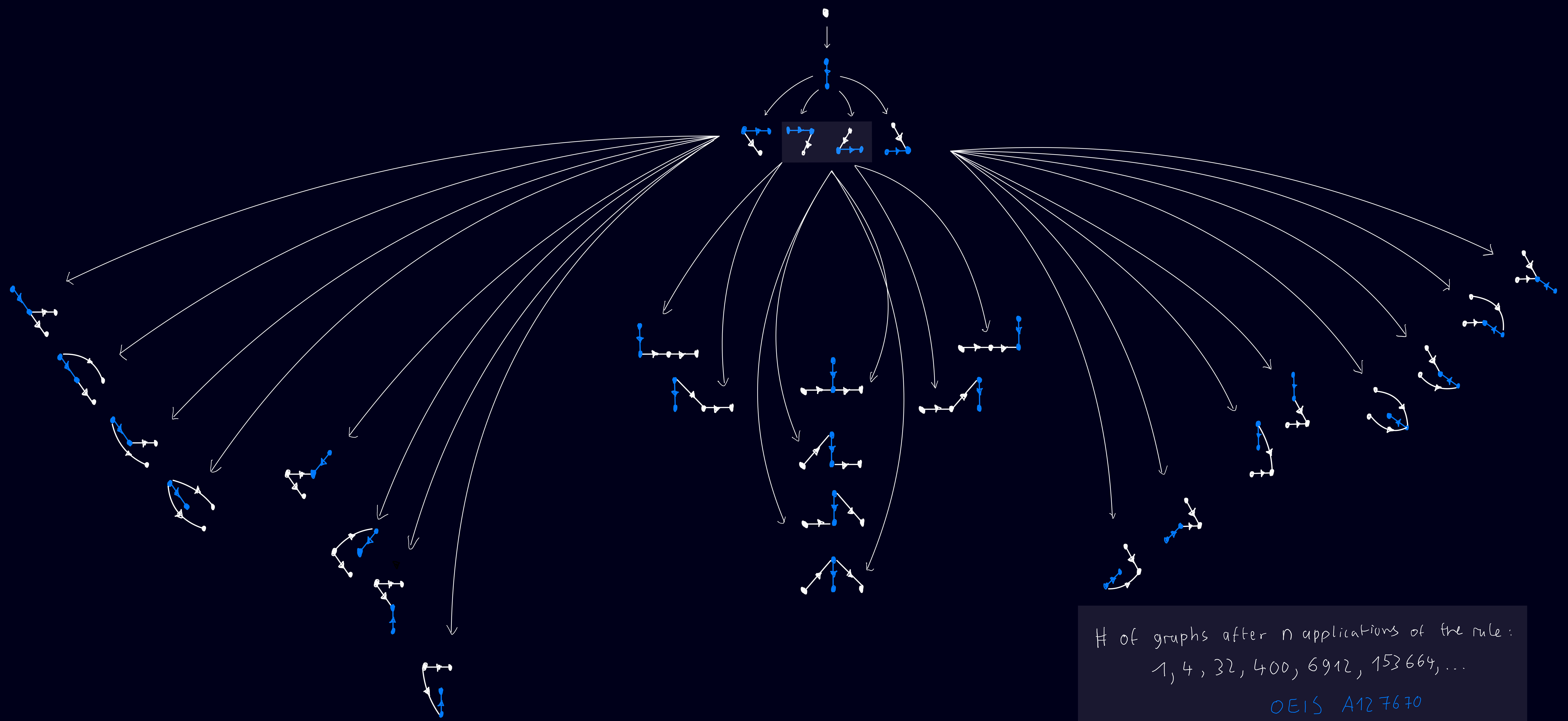
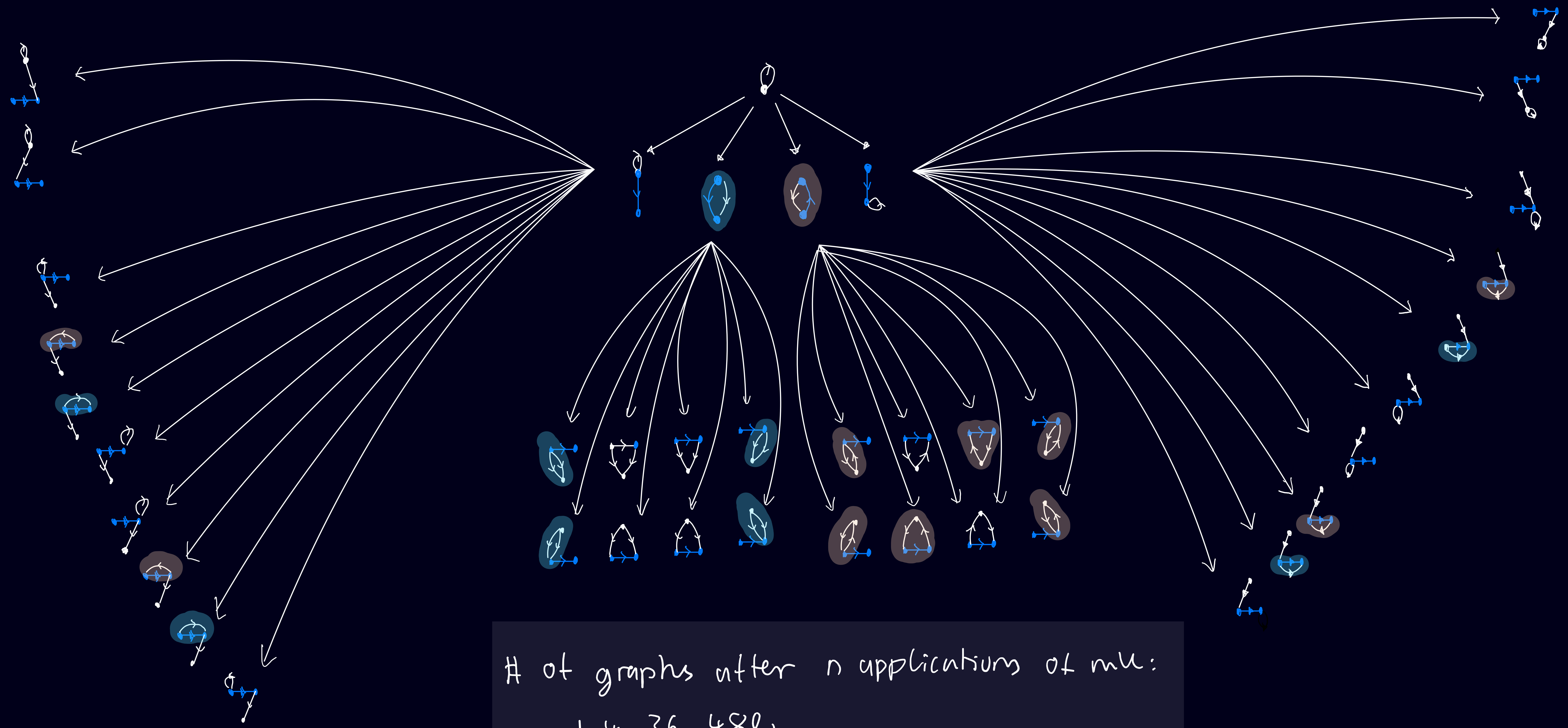


# of graphs after n applications of the rule:
1, 4, 32, 400, 6912, 153664, ...
OEIS A127670

③ Non-linear Rules in Kontsevich's Graph Cohomology Calculus



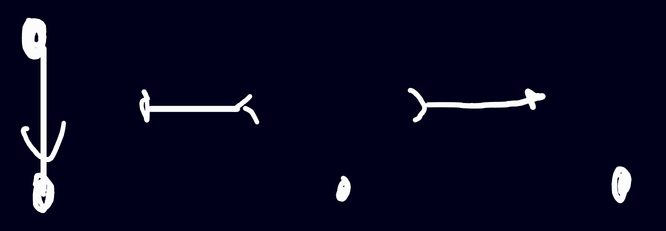
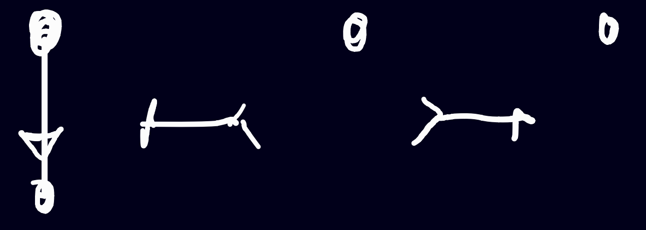
\# of graphs after n applications of mL:

1, 4, 36, 480, ...

3 NON-LINEAR RULES IN KONTSEVICH'S GRAPH COHOMOLOGY CALCULUS

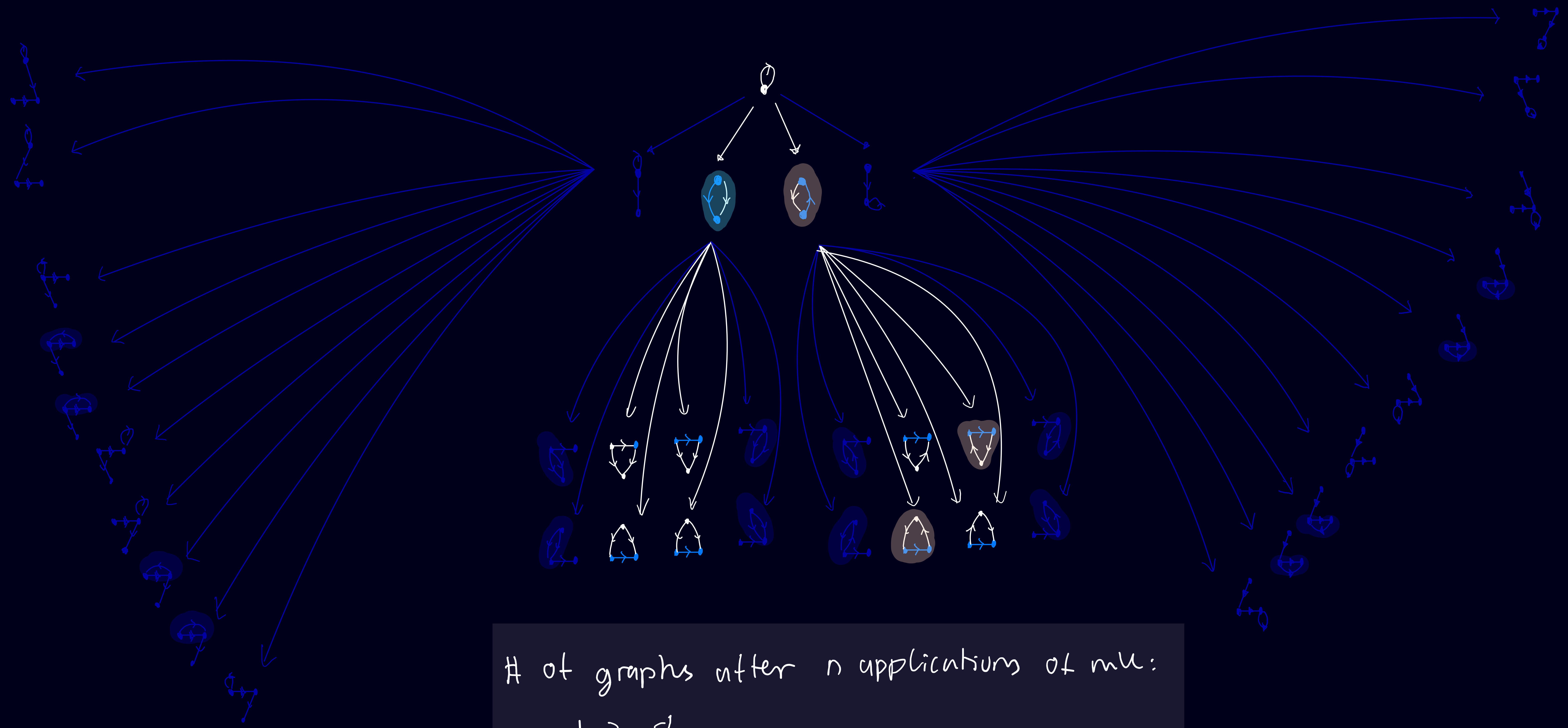
SEMANTICS RELEVANT TO KONTSEVICH'S CALCULUS:

$$\text{``}\hat{R}_K := \hat{r}_K - \hat{r}_{in} - \hat{r}_{out}\text{''}$$

$r_{in} :=$  , $r_{out} :=$ 

effectively, these generate cases where new edge is an "antenna" to the input graph state

③ Non-linear Rules in Kontsevich's Graph Cohomology Calculus



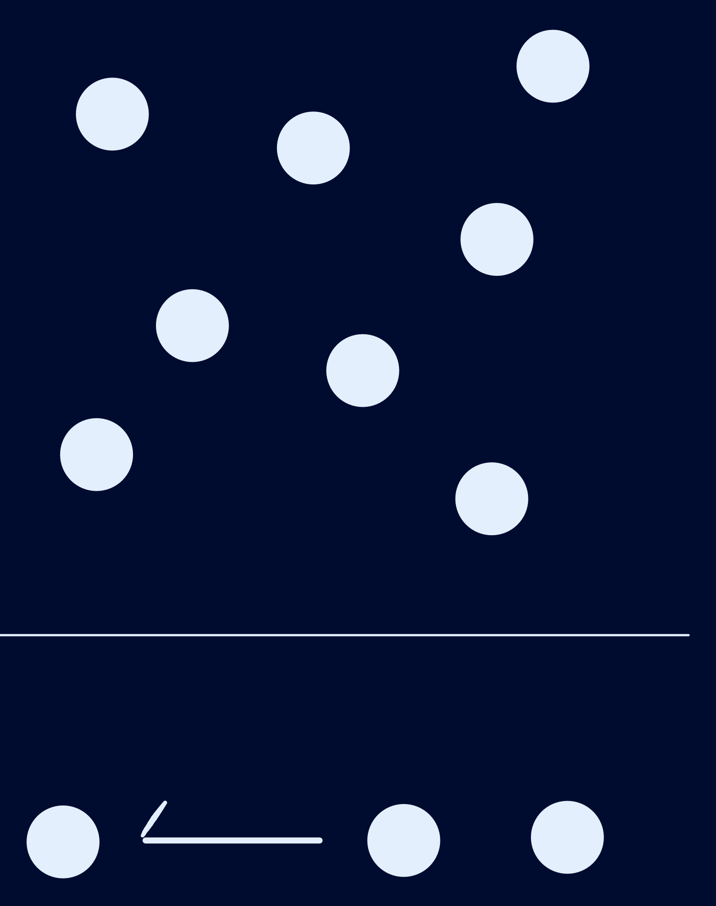
\# of graphs after n applications of mu:

1, 2, 8, ...

④ Motivation: Formalization of Rewriting + Combinatorics!

- Inspiration from combinatorial species theory:

(i) $\hat{X}(x^n) = x^{n+1}$, $\quad \frac{d}{dx}(x^n) = (n)_1\, x^{n-1} \equiv \begin{cases} 0, & n=0 \\ n x^{n-1}, & \text{else} \end{cases}$



$\hat{X}\left(\frac{d}{dx}\right)^2 x^n = (n)_2\, x^{n-1}$

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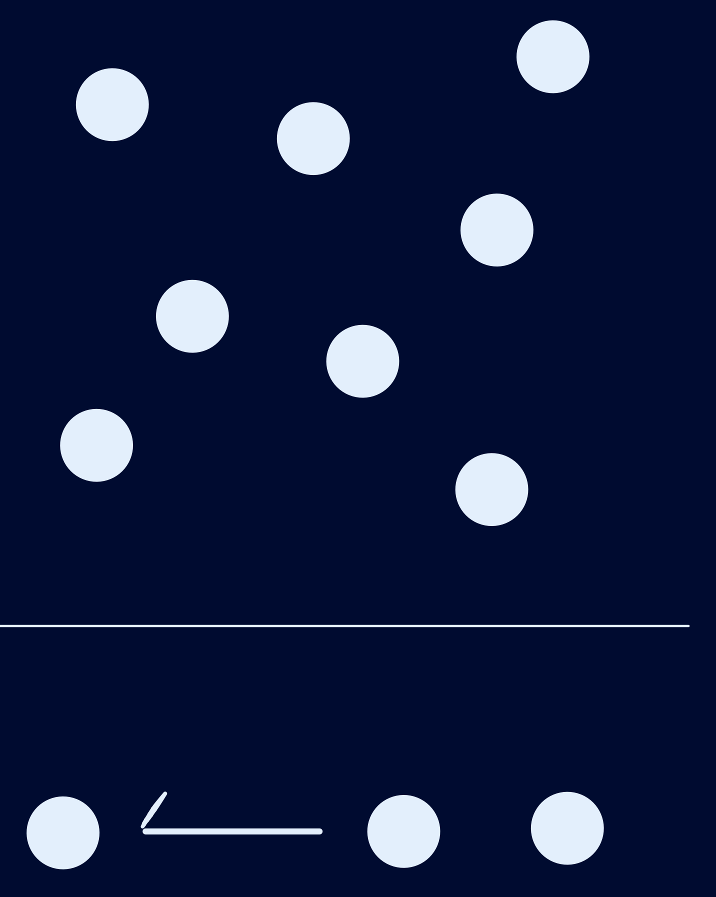
(ii) $x^n = \hat{x}^n(1)$

$$\hat{x}^p \left(\frac{d}{dx}\right)^q (x^n) = (n)_q\, x^{n-q+p}$$

\# of ways to remove q elements from a set of n elements



$$\hat{x}\left(\frac{d}{dx}\right)^2 x^n = (n)_2\, x^{n-1}$$

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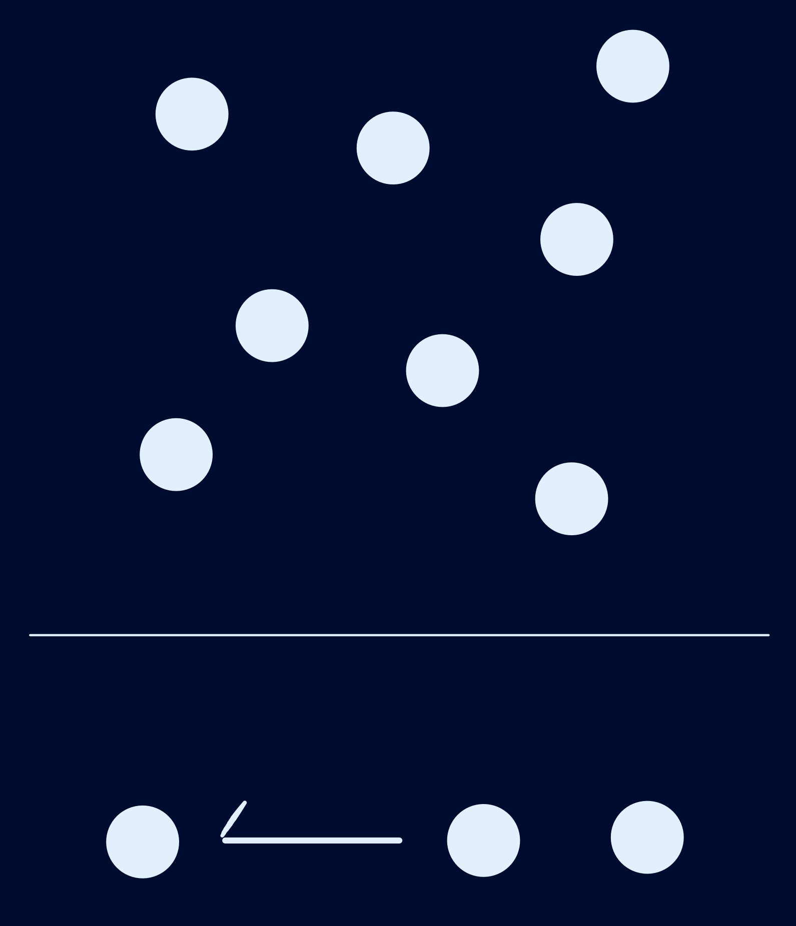

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(iii) $\hat{x}^p \left(\frac{d}{dx}\right)^q \hat{x}^r \left(\frac{d}{dx}\right)^s = \sum_{k \geq 0} \underbrace{\binom{q}{k} k! \binom{r}{k}}_{\in \mathbb{Z}_{\geq 0}} \hat{x}^{p+r-k} \left(\frac{d}{dx}\right)^{q+s-k}$

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GOAL: ANALOGA IN REWRITING THEORY

(I) "$\varrho(\delta(r))|X\rangle := \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \underbrace{\mu_X^Y}_{\in \mathbb{Z}_{\geq 0}} |Y\rangle$"

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(IV) "$\rho(\delta(r_2))\rho(\delta(r_1)) = \rho(\delta(r_2) \odot \delta(r_1))$"

$\odot$ — RULE ALGEBRA PRODUCT

④ ANSATZ: CATEGORIFICATION

GOAL:

(i) " $\varrho(\delta(r))|X\rangle := \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{y} \overbrace{\mu_X^y}^{\in \mathbb{Z}_{\geq 0}} |y\rangle$ "

(ii) " $|X\rangle = \varrho(\delta(X \leftarrow \emptyset))|\emptyset\rangle$ "

(iii) " $\varrho(\delta(r_2))\varrho(\delta(r_1)) = \sum_{r_{21}} \underbrace{\mu_{r_1, r_2}^{r_{21}}}_{\in \mathbb{Z}_{\geq 0}} \varrho(\delta(r_{21}))$ "

(iv) " $\varrho(\delta(r_2))\varrho(\delta(r_1)) = \varrho(\delta(r_2) \odot \delta(r_1))$ "

$\odot$ – RULE ALGEBRA PRODUCT

I. FORMALIZE
$$\begin{tikzcd} O \arrow[d] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d] \\ r_\alpha(X) & X \arrow[l] \end{tikzcd}$$
AS 2-CELLS IN A DOUBLE CATEGORY

II. FORMALIZE $\mathbb{Z}_{\geq 0}$-COEFFICIENTS AS CARDINALITIES (OF SUITABLE SETS...)

METHODS: DOUBLE CATEGORIES, PRESHEAVES, FIBRATIONS, COENDS, MULTISUMS ...

⑤ Definition: A Double Category $\mathbb{D}$ is a (pseudo) internal category in <u>CAT</u>

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"'] \end{tikzcd}$$

$\mathbb{D}_0$:
$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

"0-cells" — objects of $\mathbb{D}_0$

"vertical morphisms" — morphisms of $\mathbb{D}_0$

$\mathbb{D}_1$:
$$\begin{tikzcd} O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd}$$

"horizontal morphisms" — objects of $\mathbb{D}_1$

"2-cells" - morphisms of $\mathbb{D}_1$

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\end{tikzcd}$$

$$\begin{tikzcd}
O \arrow[d, "n"'] \\ Y
\end{tikzcd}
\overset{T}{\longleftarrow\!\!\!\mid}
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"' near end] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
\overset{S}{\longmapsto}
\begin{tikzcd}
I \arrow[d, "m"] \\ X
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[d, "f"'] \\ B
\end{tikzcd}
\overset{u}{\longmapsto}
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow id"' near end] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\
B & B \arrow[l, "id_B"]
\end{tikzcd}$$

$\mathbb{D}_0$: $\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$\mathbb{D}_1$: $\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"' near end] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}$

"horizontal morphisms" – objects of $\mathbb{D}_1$

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⑤ DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A (PSEUDO) INTERNAL CATEGORY IN $\underline{CAT}$

$$
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\end{tikzcd}
$$

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\begin{tikzcd}
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Y
\end{tikzcd}
$$

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Y
\end{tikzcd}
\quad \overset{T}{\longleftarrow\!\shortmid} \quad
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
\quad \overset{S}{\longmapsto} \quad
\begin{tikzcd}
I \arrow[d, "m"] \\
X
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[d, "f"'] \\
B
\end{tikzcd}
\quad \overset{u}{\longmapsto} \quad
\begin{tikzcd}
A \arrow[d, "f"'] \arrow[r, phantom, "\Downarrow id"] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\
B & B \arrow[l, "id_B"]
\end{tikzcd}
$$

$\mathbb{D}_1$:
$$
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
Y & X \arrow[l, "s"]
\end{tikzcd}
$$

"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

$$
\begin{tikzcd}
O \arrow[d, "n"'] \arrow[r, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\
O' & I' \arrow[l, "r'"]
\end{tikzcd}
\;\diamond_v\;
\begin{tikzcd}
O' \arrow[d, "n'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & I' \arrow[l, "r'"'] \arrow[d, "m'"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}
:=
\begin{tikzcd}
O \arrow[d, "n' \circ n"'] \arrow[r, phantom, "\Downarrow \alpha' \circ \alpha"] & I \arrow[l, "r"'] \arrow[d, "m' \circ m"] \\
O'' & I'' \arrow[l, "r''"]
\end{tikzcd}
$$

"vertical composition"

– composition in $\mathbb{D}_1$

⑤ DEFINITION: A DOUBLE CATEGORY $\mathbb{D}$ IS A (PSEUDO) INTERNAL CATEGORY IN $\underline{CAT}$

$$\begin{tikzcd} \mathbb{D}_0 \arrow[r, "u"] & \mathbb{D}_1 \arrow[l, bend right, "S"'] \arrow[l, bend left, "T"] & \mathbb{D}_1 \times_{\mathbb{D}_0} \mathbb{D}_1 \arrow[l, "\diamond_h"'] \end{tikzcd}$$

$\mathbb{D}_0$:
$$\begin{tikzcd} X \arrow[d, "f"] \\ Y \end{tikzcd}$$

"0-cells" – objects of $\mathbb{D}_0$

"vertical morphisms" – morphisms of $\mathbb{D}_0$

$$\begin{tikzcd} O \arrow[d, "n"'] \\ Y \end{tikzcd} \quad \overset{T}{\longmapsfrom} \quad \begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y & X \arrow[l, "s"] \end{tikzcd} \quad \overset{S}{\longmapsto} \quad \begin{tikzcd} I \arrow[d, "m"] \\ X \end{tikzcd}$$

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \quad \overset{U}{\longmapsto} \quad \begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, phantom, "\Downarrow id"] & A \arrow[l, "id_A"'] \arrow[d, "f"] \\ B & B \arrow[l, "id_B"] \end{tikzcd}$$

$\mathbb{D}_1$:
$$\begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ Y & X \arrow[l, "s"] \end{tikzcd}$$

"horizontal morphisms" – objects of $\mathbb{D}_1$

"2-cells" – morphisms of $\mathbb{D}_1$

$$\begin{tikzcd} O \arrow[d, "n"'] \arrow[dr, phantom, "\Downarrow\alpha"] & I \arrow[l, "r"'] \arrow[d, "m"] \\ O' & I' \arrow[l, "r'"] \end{tikzcd} \;\; \diamond_v \;\; \begin{tikzcd} O' \arrow[d, "n'"'] \arrow[dr, phantom, "\Downarrow\alpha'"] & I' \arrow[l, "r'"'] \arrow[d, "m'"] \\ O'' & I'' \arrow[l, "r''"] \end{tikzcd} \;:=\; \begin{tikzcd} O \arrow[d, "n' \circ n"'] \arrow[dr, phantom, "\Downarrow\alpha' \circ \alpha"] & I \arrow[l, "r"'] \arrow[d, "m' \circ m"] \\ O'' & I'' \arrow[l, "r''"] \end{tikzcd}$$

"vertical composition"
— composition in $\mathbb{D}_1$

$$\begin{tikzcd} C \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\alpha_2"] & B \arrow[l, "r_2"'] \arrow[d, "b"] \\ C' & B' \arrow[l, "s_2"] \end{tikzcd} \;\; \diamond_h \;\; \begin{tikzcd} B \arrow[d, "b"'] \arrow[dr, phantom, "\Downarrow\alpha_1"] & A \arrow[l, "r_1"'] \arrow[d, "a"] \\ B' & A' \arrow[l, "s_1"] \end{tikzcd} \;:=\; \begin{tikzcd} C \arrow[d, "c"'] \arrow[dr, phantom, "\Downarrow\alpha_2 \diamond_h \alpha_1"] & A \arrow[l, "r_2 \diamond_h r_1"'] \arrow[d, "a"] \\ C' & A' \arrow[l, "s_2 \diamond_h s_1"] \end{tikzcd}$$

"horizontal composition" ← IN GENERAL ONLY WEAKLY ASSOCIATIVE!

6 DEFINITION: A **PRESENTATION** OF A DOUBLE CATEGORY $\mathbb{D}$ IS A FAMILY $(h_n)_{n \geq 0}$ OF FUNCTORS $h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$, WHERE $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

$$h_0 := \mathbb{U}, \quad h_1 := id, \quad h_2(-_2, -_1) := -_2 \diamond_h -_1,$$

$$\forall n \geq 2 : \; h_{n+1}(-_{n+1}, \ldots, -_1) \cong h_2(-_{n+1}, h_n(-_n, \ldots, -_1))$$

- NOTATIONAL CONVENTION:

$$\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l, "r_n"'] & \cdots & \cdot & \cdot \arrow[l, "r_1"'] \arrow[d] \\
\cdot & & \Downarrow\sigma & & \cdot \arrow[llll, "s"]
\end{tikzcd}
\equiv
\begin{tikzcd}
\cdot \arrow[d] & \cdot \arrow[l, "r_n"'] & \cdots & \cdot & \cdot \arrow[l, "r_1"'] \arrow[d] \arrow[llll, bend left, "h_n(r_n, \ldots, r_1)"] \\
\cdot & & \Downarrow\sigma & & \cdot \arrow[llll, bend left, "s"]
\end{tikzcd}$$

⑥ DEFINITION: A PRESENTATION OF A DOUBLE CATEGORY $\mathbb{D}$ IS A FAMILY $(h_n)_{n\geq 0}$

OF FUNCTORS $h_n : \mathbb{D}_n \to \mathbb{D}_1$, WHERE $\mathbb{D}_n := \underbrace{\mathbb{D}_1 \times_{\mathbb{D}_0} \cdots \times_{\mathbb{D}_0} \mathbb{D}_1}_{n \text{ times}}$,

$$h_0 := U, \quad h_1 := \mathrm{id}, \quad h_2(-_2, -_1) := -_2 \diamond_h -_1,$$

$$\forall n \geq 2: \; h_{n+1}(-_{n+1}, \ldots, -_1) \cong h_2(-_{n+1}, h_n(-_n, \ldots, -_1))$$

- NOTATIONAL CONVENTION:

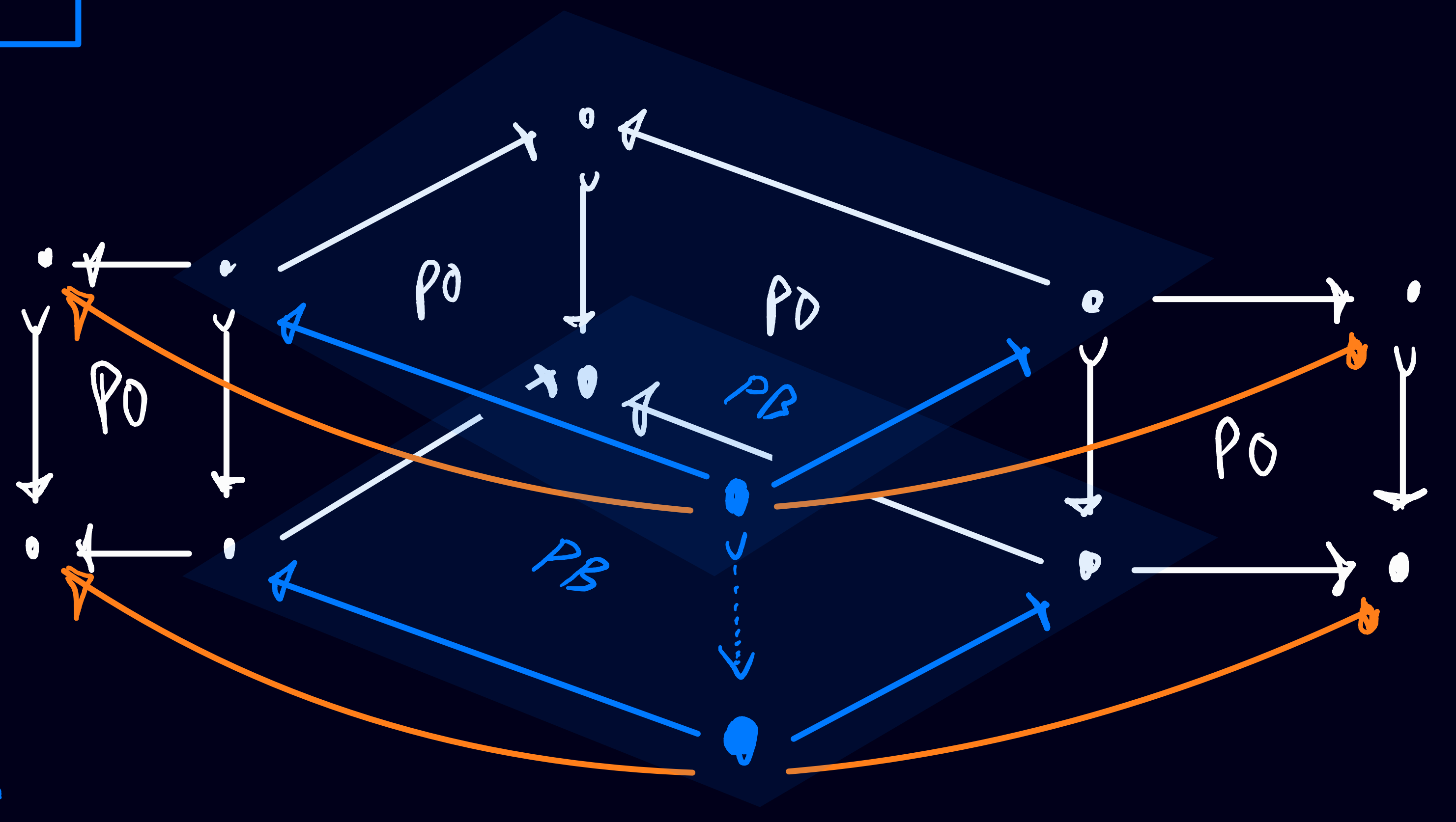
$$\boxed{\begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "r_n"'] & \cdots & \bullet & \bullet \arrow[l, "r_1"'] \arrow[d] \\ \bullet & & \Downarrow\sigma & & \bullet \arrow[llll, "s"] \end{tikzcd} \quad \equiv \quad \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "r_n"'] & \cdots & \bullet & \bullet \arrow[l, "r_1"'] \arrow[d] \arrow[llll, bend left, "h_n(r_n,\ldots,r_1)"] \\ \bullet & & \Downarrow\sigma & & \bullet \arrow[llll, bend left, "s"] \end{tikzcd}}$$

- EXAMPLE:

$$\begin{tikzcd} O \arrow[d, "n"'] & I \arrow[l, "r"'] \arrow[d, "m"] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd} := \begin{tikzcd} O \arrow[d, "n"'] & K_r \arrow[l, "o_r"'] \arrow[r, "i_r"] \arrow[d, "k_\alpha"] \arrow[dl, phantom, "PO"] & I \arrow[d, "m"] \arrow[dl, phantom, "PO"] \\ Y & K_s \arrow[l, "o_s"] \arrow[r, "i_s"'] & X \end{tikzcd}$$

↗ HORIZONTAL COMPOSITION:

≙ CHOICE OF PULLBACKS (PBs)!



⑦ KEY CONCEPT: (COVARIANT) PRESHEAVES $F:\mathbb{D}_1\to \underline{\mathbf{Set}}$

↳ IDEA: $\forall r\in\mathbb{D}_1:\quad \hat{\Delta}_r := \mathbb{D}_1(r,-)$

↳ $$\left|\hat{\Delta}_r\left(Y \xleftarrow{s} X\right)\right| = \left|\left\{ \begin{array}{ccc} O & \xleftarrow{r} & I \\ {\scriptstyle n}\downarrow & \Downarrow\alpha & \downarrow{\scriptstyle m} \\ Y & \xleftarrow[s]{} & X\end{array} \in \mathbb{D}_1\right\}\right| \propto$$ "# ways to rewrite X into Y along $Y\xleftarrow{s}X$ with rule $O\xleftarrow{r}I$"

▸ BUT: we want "$\rho(\delta(r))|X\rangle = \underbrace{\rho(\delta(r))\,\rho(\delta(\overbrace{X\leftarrow\emptyset}^{?}))\,|\overset{?}{\emptyset}\rangle}_{?} = \sum_\alpha |r_\alpha(X)\rangle = \sum_Y \underbrace{\overbrace{M_{r,X}^{Y}}^{\in\mathbb{Z}_{\geq 0}}}_{?}|Y\rangle$"

⑦ KEY CONCEPT: (COVARIANT) PRESHEAVES $F : \mathbb{D}_1 \to \underline{\mathbf{Set}}$

↳ <u>IDEA</u>: $\forall r \in \mathbb{D}_1 : \quad \hat{\Delta}_r := \mathbb{D}_1(r, -)$

↳ $\left|\hat{\Delta}_r\left(Y \xleftarrow{s} X\right)\right| = \left|\left\{\begin{array}{ccc} O & \xleftarrow{r} & I \\ n\downarrow & \Downarrow\alpha & \downarrow m \\ Y & \xleftarrow{s} & X \end{array} \in \mathbb{D}_1\right\}\right| \propto$ "# ways to rewrite X into Y along $Y \xleftarrow{s} X$ with rule $O \xleftarrow{r} I$"

- BUT: we want "$\underbrace{\rho(\delta(r))\,\rho(\delta(\overbrace{X \leftarrow \emptyset}^{?}))\,|\overset{?}{\emptyset}\rangle}_{?}$" — in full: "$\rho(\delta(r))\,|X\rangle = \underbrace{\rho(\delta(r))\,\rho(\delta(\overbrace{X \leftarrow \emptyset}^{?}))\,|\overset{?}{\emptyset}\rangle}_{?} = \sum_{\alpha} |r_\alpha(X)\rangle = \sum_{Y} \underbrace{\overbrace{M_{r,X}^{Y}}^{\in \mathbb{Z}_{\geq 0}}}_{?} |Y\rangle$"

- <u>ASSUMPTION</u>: $\mathbb{D}_0$ HAS A STRICT INITIAL OBJECT $\emptyset$ (i.e., $\forall X \in \mathbb{D}_0 : \exists!\, \emptyset \rightarrowtail X \wedge \forall X \xrightarrow{f} \emptyset : X = \emptyset$),

AND SUCH THAT (i) $\forall X \in \mathbb{D}_0 : \exists!\, (X \leftarrow \emptyset) \in ob(\mathbb{D}_1) \wedge \exists!\, (\emptyset \leftarrow X) \in ob(\mathbb{D}_1)$

(ii) $\forall \begin{array}{c} X \\ f\downarrow \\ Y \end{array} \in \mathbb{D}_0 : \left|\left\{\begin{array}{ccc} X & \leftarrow & \emptyset \\ f\downarrow & \Downarrow\alpha & \| \\ Y & \leftarrow & \emptyset \end{array} \in \mathbb{D}_1\right\}\right| \leq 1 \wedge \left|\left\{\begin{array}{ccc} \emptyset & \leftarrow & X \\ \| & \Downarrow\beta & \downarrow f \\ \emptyset & \leftarrow & Y \end{array} \in \mathbb{D}_1\right\}\right| \leq 1$

⑧ DEFINITION: A COEND FOR A FUNCTOR $F : \mathcal{C}^{op} \times \mathcal{C} \to \underline{\mathbf{Set}}$

IS DEFINED AS $\displaystyle\int^{C \in \mathcal{C}} F(C,C) = \left(\coprod_{C \in \mathcal{C}} F(C,C) \right) \Big/ \sim$

$$\underset{F(C,C)}{\Downarrow} \qquad \underset{F(C',C')}{\Downarrow}$$

with: $(C, x) \sim (C', x') :\Longleftrightarrow \exists\, C \xrightarrow{\gamma} C',\ y \in F(C', C) : x = F(\gamma, id)y \ \wedge\ x' = F(id, \gamma)y$

⑧ DEFINITION: A COEND FOR A FUNCTOR $F:\mathcal{C}^{op}\times\mathcal{C}\to\underline{\mathbf{Set}}$

IS DEFINED AS

$$\int^{C\in\mathcal{C}} F(C,C) = \left(\coprod_{C\in\mathcal{C}} F(C,C)\right)\Big/\sim$$

with: $(C,\overset{\substack{F(C,C)\\ \cup}}{x})\sim(C',\overset{\substack{F(C',C')\\ \cup}}{x'}) :\Leftrightarrow \exists\, C\xrightarrow{\gamma}C',\ y\in F(C',C) : x=F(\gamma,\mathrm{id})y \ \wedge\ x'=F(\mathrm{id},\gamma)y$

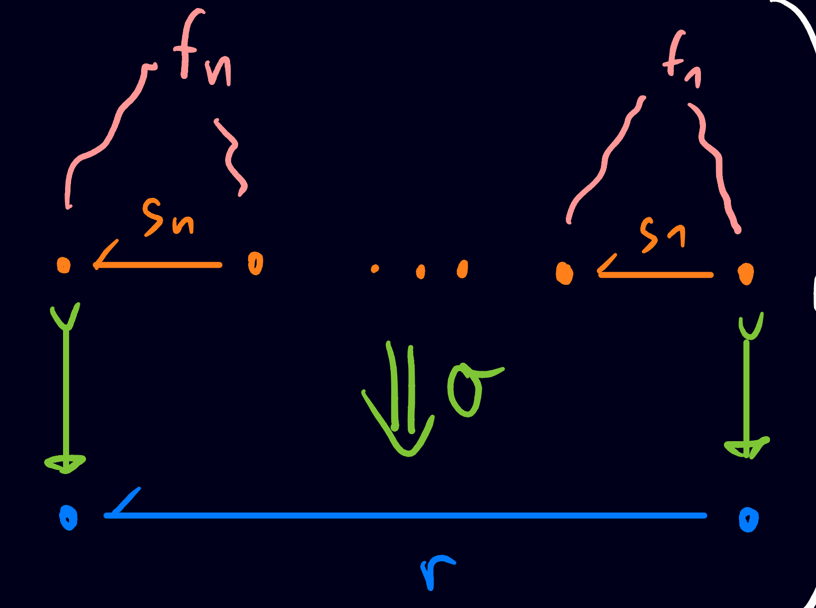
KEY CONCEPT: CONVOLUTION PRODUCTS OF PRESHEAVES $F_n,\dots,F_1:\mathbb{D}_1\to\underline{\mathbf{Set}}$

$$(F_n*\cdots*F_1) := r\mapsto \int^{S=(S_n,\dots,S_1)\in\mathbb{D}_n}\mathbb{D}_1(h_n(S),r)\times\mathbb{F}_n(S)$$

$(\cong \mathrm{Lan}_{h_n}(\mathbb{F}_n))$

- $\mathbb{D}_1(h_n(-),r):\mathbb{D}_n^{op}\to\underline{\mathbf{Set}}$
- $\mathbb{F}_n := F_n\times\dots\times F_1:\mathbb{D}_n\to\underline{\mathbf{Set}}$

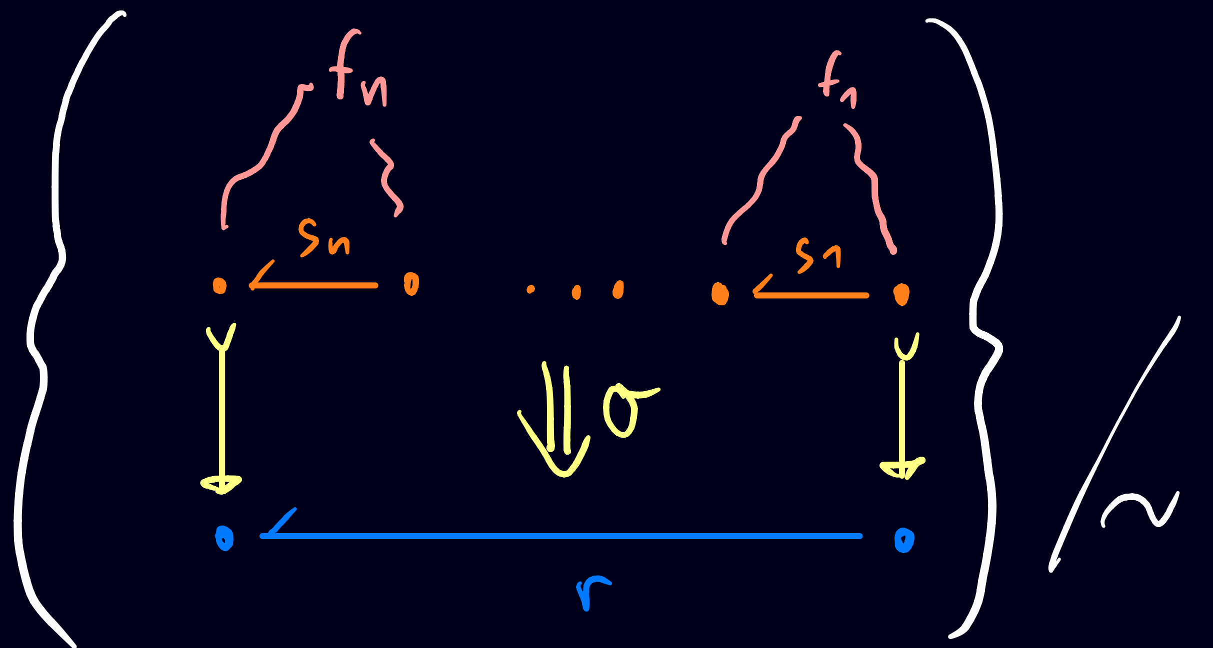
$$= \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S\in\mathbb{D}_n\\ \sigma\in\mathbb{D}_1(h_n(S),r)\\ f\in\mathbb{F}_n(S)\end{array}\right\}\Big/\sim \;\equiv\; \left\{ \right.$$



$\left.\right\}\Big/\sim$

9

$$(\mathbb{F}_n * \dots * \mathbb{F}_1)(r) = \left\{ (S,(\sigma,f)) \;\middle|\; \begin{array}{l} S\in \mathbb{D}_n\\ \sigma \in \mathbb{D}_1(h_n(S), r)\\ f\in \mathbb{F}_n(S)\end{array}\right\}\Big/_{\sim} \equiv$$

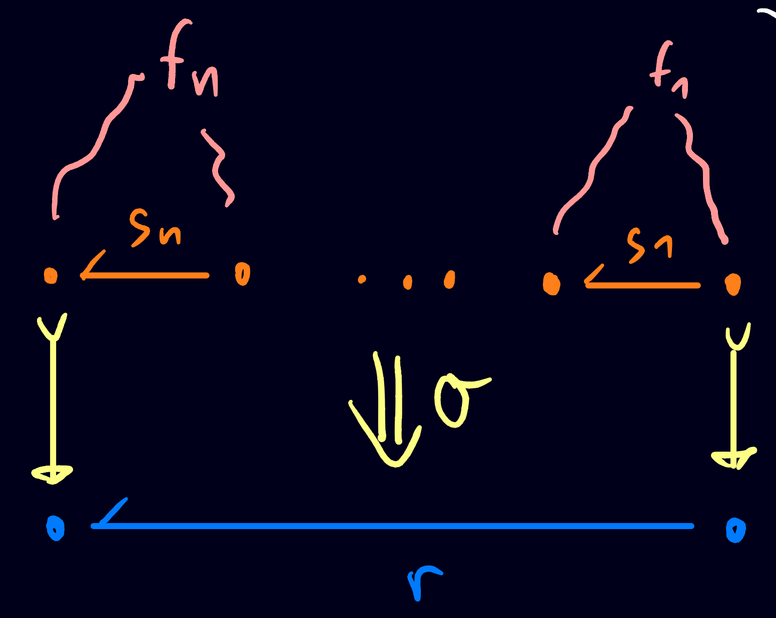


- $(S,(\sigma,f)) \sim (S',(\sigma',f')) :\Leftrightarrow \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau,g)\in \mathbb{D}_1(h_n(S'),r)\times \mathbb{F}_n(S):$

$$(\sigma,f) = (\mathbb{D}_1(h_n(A),r)\tau,\ g) \;\wedge\; (\sigma',f') = (\tau,\ \mathbb{F}_n(A)g)$$

9

$$(\mathbb{F}_n * \dots * \mathbb{F}_1)(r) = \left\{ (S, (\sigma, f)) \;\middle|\; \begin{array}{l} S \in \mathbb{D}_n \\ \sigma \in \mathbb{D}_1(h_n(S), r) \\ f \in \mathbb{F}_n(S) \end{array} \right\} \Big/ \sim \;\; \equiv \left\{ \right.$$

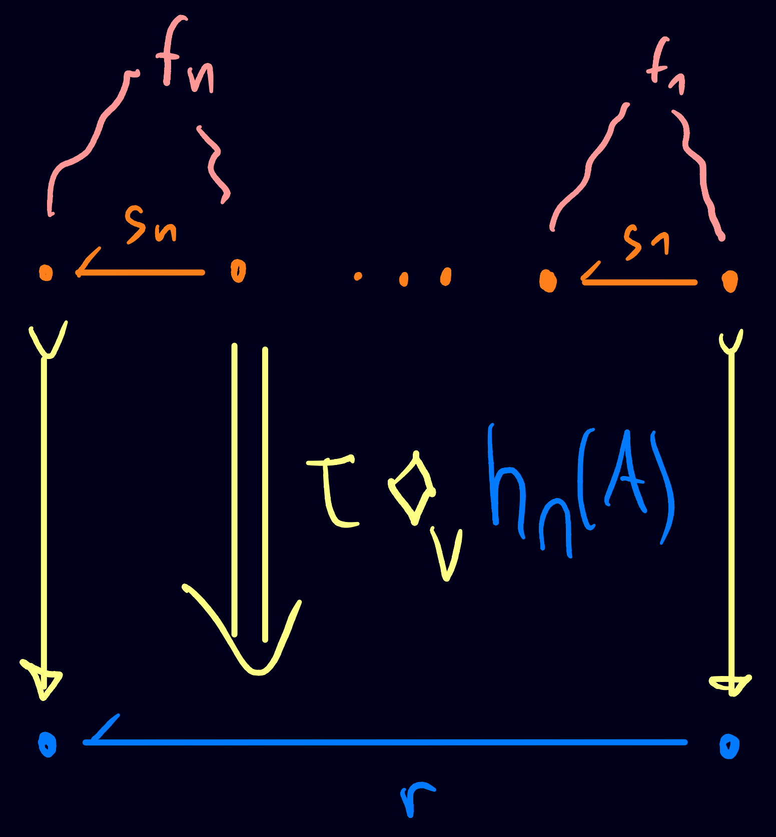


$$\left. \right\} \Big/ \sim$$

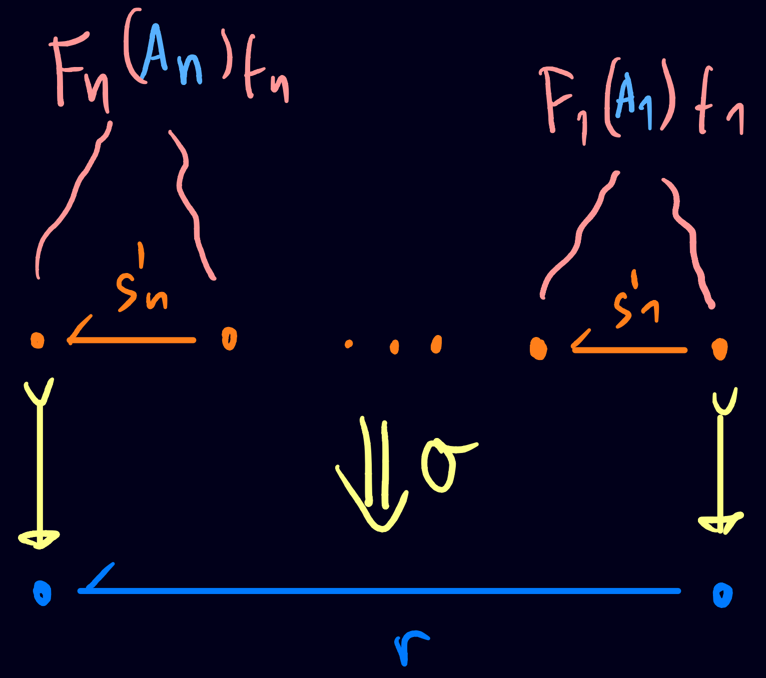
- $(S,(\sigma,f)) \sim (S',(\sigma',f')) :\Leftrightarrow \exists\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ (\tau, g) \in \mathbb{D}_1(h_n(S'), r) \times \mathbb{F}_n(S):$

$$(\sigma, f) = (\mathbb{D}_1(h_n(A), r)\tau,\ g) \;\wedge\; (\sigma', f') = (\tau,\ \mathbb{F}_n(A) g)$$

↳ $\forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,$
$h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:

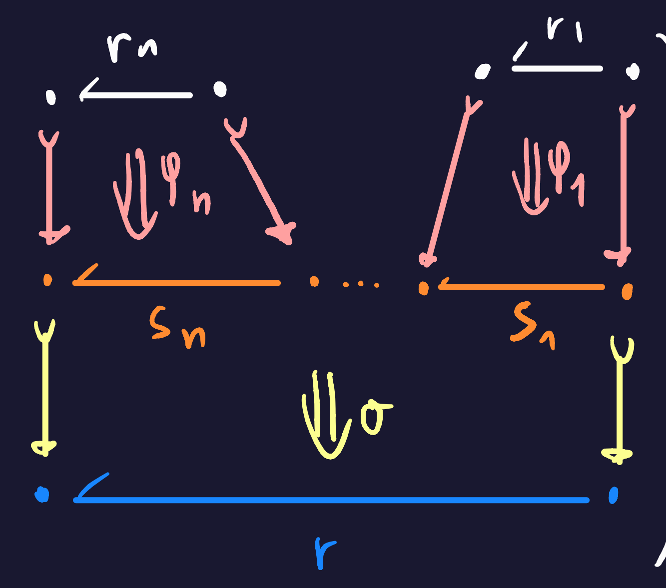


$\sim$



EXAMPLE: $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -) : \mathbb{D}_1 \to \underline{Set}$ $\quad (j = 1, \dots, n)$

↳ $(\hat{\Delta}_{r_n} * \dots * \hat{\Delta}_{r_1})(r) = \left\{ \right.$



$\left. \right\} / \sim$

10 Key concept: Fibrational structures

- Definition: A functor $G:\mathcal{E}\to\mathcal{B}$ is a Grothendieck opfibration iff

$$
\forall\;
\begin{tikzcd}
e \arrow[d, mapsto, "G"'] & \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\;:\;
\exists\;
\begin{tikzcd}
e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] & e' \arrow[d, mapsto, "G"] \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\;:\;
\forall\;
\begin{tikzcd}
e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, mapsto, "G"'] & e'' \arrow[d, mapsto, "G"] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b''
\end{tikzcd}
\;:\;
\begin{tikzcd}
e \arrow[r, "\gamma(f)"] \arrow[d, mapsto, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, mapsto, "G"'] \arrow[r, dashed, "\beta", "\exists!"'] & e'' \arrow[d, mapsto, "G"] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=G(\beta)"'] & b''
\end{tikzcd}
$$

⑩ Key concept: Fibrational structures

- **Definition:** A functor $G:\mathcal{E}\to\mathcal{B}$ is a **Grothendieck opfibration** iff

$$\forall\ \begin{tikzcd} e \arrow[d, "G"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd} \ :\ \exists\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] & e' \arrow[d, "G"] \\ b \arrow[r, "f"'] & b' \end{tikzcd} \ :\ \forall\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, "G"'] & e'' \arrow[d, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd} \ :\ \begin{tikzcd} e \arrow[r, "\gamma(f)"] \arrow[d, "G"'] \arrow[rr, bend left, "\varepsilon"] & e' \arrow[d, "G"'] \arrow[r, dashed, "\beta"', "\exists!"] & e'' \arrow[d, "G"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "G(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=G(\beta)"'] & b'' \end{tikzcd}$$

- **Definition:** A functor $S:\mathcal{E}\to\mathcal{B}$ is a **Street opfibration** iff

$$\forall\ \begin{tikzcd} e \arrow[d, "S"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd} \ :\ \exists\ \begin{tikzcd} e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] & & \tilde{e} \arrow[d, "S"] \\ b \arrow[r, "f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \end{tikzcd} \ :\ \forall\ \begin{tikzcd} e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] \arrow[rrr, bend left, "\varepsilon"] & & \tilde{e} \arrow[d, "S"] & e'' \arrow[d, "S"] \\ b \arrow[r, "f"'] \arrow[rrr, bend right, "S(\varepsilon)=g\circ\varphi_f\circ f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \arrow[r, "g"'] & b'' \end{tikzcd} \ :\ \begin{tikzcd} e \arrow[rr, "\sigma(f)"] \arrow[d, "S"'] \arrow[rrr, bend left, "\varepsilon"] & & \tilde{e} \arrow[d, "S"] \arrow[r, dashed, "\beta"] & e'' \arrow[d, "S"] \\ b \arrow[r, "f"'] \arrow[rrr, bend right, "S(\varepsilon)=g\circ\varphi_f\circ f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \arrow[r, "g=S(\beta)"'] & b'' \end{tikzcd}$$

⑪ DEFINITION: A FUNCTOR $M:\mathcal{E}\to\mathcal{B}$ IS A MULTI-OPFIBRATION IFF

$$\forall \begin{tikzcd} e \arrow[d, maps to, "M"'] & \\ b \arrow[r, "f"'] & b \end{tikzcd} : \exists \left\{ \begin{tikzcd} e \arrow[r, "\mu_j(f)"] \arrow[d, maps to, "M"'] & e_j \arrow[d, maps to, "M"] \\ b \arrow[r, "f"'] & b' \end{tikzcd} \right\}_{j\in J_{e,f}} : \forall \begin{tikzcd} e \arrow[rr, bend left, "\varepsilon"] \arrow[d, maps to, "M"'] & & e'' \arrow[d, maps to, "M"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd} : \exists j\in J_{e,f} : \begin{tikzcd} e \arrow[r, "\mu_j(f)"'] \arrow[rr, bend left, "\varepsilon"] \arrow[d, maps to, "M"'] & e'_j \arrow[r, dashed, "\beta_j", "\exists!"'] \arrow[d, maps to, "M"] & e'' \arrow[d, maps to, "M"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g=M(\beta_j)"'] & b'' \end{tikzcd}$$

$$\wedge\ \forall \begin{tikzcd} e \arrow[rr, bend left, "\varepsilon"] \arrow[r, "x"] \arrow[d, maps to, "M"'] & X \arrow[r, "y"] \arrow[d, maps to, "M"] & e'' \arrow[d, maps to, "M"] \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd} : \begin{tikzcd} e \arrow[rr, bend left, "\varepsilon"] \arrow[r, "\mu_j(f)"'] \arrow[dr, bend right, "x"'] \arrow[dd, maps to, "M"'] & e'_j \arrow[r, dashed, "\beta_j"'] \arrow[d, dashed, "\exists!"] \arrow[dd, maps to, bend right, "M"'] & e'' \arrow[dd, maps to, "M"] \\ & X \arrow[ur, "y"'] \arrow[d, maps to, "M"] & \\ b \arrow[r, "f"'] \arrow[rr, bend right, "M(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b'' \end{tikzcd}$$

12 DEFINITION: A FUNCTOR $R:\mathcal{E}\to\mathcal{B}$ IS A RESIDUAL MULTI-OPFIBRATION IFF

$$
\forall\;
\begin{tikzcd}
e \arrow[d, mapsto, "R"'] & \\
b \arrow[r, "f"'] & b'
\end{tikzcd}
\;:\;\exists\;
\left\{
\begin{tikzcd}
e \arrow[r, "\rho_j(f)"] \arrow[d, mapsto, "R"'] & e'_j \arrow[d, mapsto] & \\
b \arrow[r, "f"'] & b' \arrow[r, "f_{x_j}"'] & b'_j
\end{tikzcd}
\right\}_{j\in J_{e,f}}
$$

"residue"

$$
:\;\forall\;
\begin{tikzcd}
e \arrow[rr, bend left, "\varepsilon"] \arrow[d, mapsto, "R"'] & & e'' \arrow[d, mapsto] \\
b \arrow[r, "f"'] \arrow[rr, bend right, "R(\varepsilon)=g\circ f"'] & b' \arrow[r, "g"'] & b''
\end{tikzcd}
\;:\;\exists j\in J_{e,f}:\;
\begin{tikzcd}
e \arrow[rr, "\rho_j(f)"] \arrow[rrr, bend left, "\varepsilon"] \arrow[ddd, mapsto, "R"'] & & e'_j \arrow[r, dashed, "\exists!\,\beta_j"] \arrow[d, mapsto, "R"] & e'' \arrow[ddd, mapsto, "R"] \\
& & b'_j \arrow[ddr, "h'_j"] & \\
& & & \\
b \arrow[r, "f"'] \arrow[rrr, bend right, "R(\varepsilon)=g\circ f"'] & b' \arrow[uur, "f_{x_j}"] \arrow[rr, "g"'] & & b''
\end{tikzcd}
$$

$$
\wedge\;\forall\;
\begin{tikzcd}
e \arrow[rr, "\chi"] \arrow[rrr, bend left, "\varepsilon=\chi\circ\gamma"] \arrow[ddd, mapsto, "R"'] & & \tilde{e} \arrow[r, "\gamma"] \arrow[d, mapsto, "R"] & e'' \arrow[ddd, mapsto, "R"] \\
& & c \arrow[ddr, "y"] & \\
& & & \\
b \arrow[r, "f"'] \arrow[rrr, bend right, "R(\varepsilon)=g\circ f"'] & b' \arrow[uur, "x"] \arrow[rr, "g"'] & & b''
\end{tikzcd}
\;:\;\exists!\; e'_j \xrightarrow{\varphi_j} \tilde{e}\;:\;
\begin{tikzcd}
e \arrow[r, "\rho_j(f)"] \arrow[rr, bend right, "\chi"'] \arrow[rrr, bend left, "\varepsilon"] \arrow[ddd, mapsto, "R"'] & e'_j \arrow[rr, dashed, "\beta_j"] \arrow[d, mapsto, "R"] \arrow[dr, "\varphi_j"] & & e'' \arrow[ddd, mapsto, "R"] \\
& b'_j \arrow[ddr, "R(\varphi_j)"'] & \tilde{e} \arrow[ur, "\gamma"] \arrow[dd, mapsto, "R"] & \\
& & & \\
b \arrow[r, "f"'] \arrow[rrr, bend right, "R(\varepsilon)=g\circ f"'] & b' \arrow[uu, "f_{x_j}"] \arrow[rr, "g"'] \arrow[r, bend right, "x"'] & c \arrow[r, "y"'] & b''
\end{tikzcd}
$$

13 DEFINITION: LET $X:\mathcal{E}\to\mathcal{B}$ BE AN X-OPFIBRATION ($X\in\{G,S,M,R\}$).

THEN A CLEAVAGE FOR X IS DEFINED AS A CHOICE OF REPRESENTATIVE FOR EACH X-OPCARTESIAN LIFTING:

$$G^*\left(\begin{tikzcd} e \arrow[d, "G"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \begin{tikzcd} e \arrow[r, "G^*(f)"] \arrow[d, "G"'] & b \arrow[d, "G"] \\ b \arrow[r, "f"'] & b \end{tikzcd}$$

$$S^*\left(\begin{tikzcd} e \arrow[d, "S"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \begin{tikzcd} e \arrow[rr, "\sigma^*(f)"] \arrow[d, "S"'] & & \tilde{e} \arrow[d, "S"] \\ b \arrow[r, "f"'] & b' \arrow[r, "\cong", "\varphi_f"'] & \tilde{b} \end{tikzcd}$$

$$M^*\left(\begin{tikzcd} e \arrow[d, "M"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \left\{\begin{tikzcd} e \arrow[r, "\mu_j^*(f)"] \arrow[d, "M"'] & e'_j \arrow[d, "M"] \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right\}_{j\in\mathfrak{Z}^*_{e;f}}$$

$$R^*\left(\begin{tikzcd} e \arrow[d, "R"'] & \\ b \arrow[r, "f"'] & b' \end{tikzcd}\right) := \left\{\begin{tikzcd} e \arrow[rr, "\rho_j^*(f)"] \arrow[d, "R"'] & & e'_j \arrow[d, "R"] \\ b \arrow[r, "f"'] & b' \arrow[r, "f x_j"'] & b'_j \end{tikzcd}\right\}_{j\in\mathfrak{Z}^*_{e;f}}$$

ONE REPRESENTATIVE PER EQUIVALENCE CLASS IN $\mathfrak{Z}_{e;f}$!

(14) EMPIRICAL RESULT: $\mathbb{D}$ FOR COMPOSITIONAL* REWRITING SEMANTICS

*2204.07175

- $h_2 = \diamond_h : \mathbb{D}_2 \longrightarrow \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION, i.e.,

$$\forall \quad \begin{tikzcd} R=(r_2,r_1) \arrow[d, "h_2"'] & \\ r \arrow[r, "\alpha"'] & s \end{tikzcd} \quad : \exists \quad \begin{tikzcd} R \arrow[rr, "A"] \arrow[d, "h_2"'] & & T \arrow[d, "h_2"] \\ r \arrow[r, "\alpha"'] & s \arrow[r, "\cong", "\varphi_\alpha"'] & t \end{tikzcd} \quad : \; S(\varphi_\alpha) = id_{S(s)} \wedge T(\varphi_\alpha) = id_{T(s)} \wedge$$

(STREET OPFIBRATION CONDITIONS)

$$\forall \quad \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l, "r_2"'] \arrow[d, phantom, "\Downarrow\alpha"] & \bullet \arrow[l, "r_1"'] \arrow[d] \\ \bullet & {} & \bullet \arrow[ll, "s"] \end{tikzcd} \quad : \exists \quad \begin{tikzcd} \bullet \arrow[d] \arrow[dr, phantom, "\Downarrow A_2"] & \bullet \arrow[l, "r_2"'] \arrow[d] \arrow[dr, phantom, "\Downarrow A_1"] & \bullet \arrow[l, "r_1"'] \arrow[d] \\ \bullet & \bullet \arrow[l, "t_2"'] & \bullet \arrow[l, "t_1"'] \arrow[ll, bend left=50, "s"] \arrow[ll, phantom, bend left=25, "\cong \Downarrow \varphi_\alpha^{-1}"] \end{tikzcd} \quad : \; \varphi_\alpha^{-1} \diamond_v (A_2 \diamond A_1) = \alpha$$

"GLOBULAR" ISOMORPHISM

BY INDUCTION ON n, ONE FINDS THAT

$\forall n \geq 2: \; h_n : \mathbb{D}_n \longrightarrow \mathbb{D}_1$ ARE "GLOBULAR" STREET OPFIBRATIONS

15

$\mathbb{D}_0$ HAS MULTI-SUMS:

$$\forall\ (A,B)\in \mathbb{D}_0\times\mathbb{D}_0 :\ \exists \left\{ A \overset{a_j}{\rightarrowtail} M_j \overset{b_j}{\leftarrowtail} B \right\}_{j\in J_{(A,B)}} :\ \forall \begin{tikzcd} A \arrow[dr, tail, bend right, "x_A"'] & & B \arrow[dl, tail, bend left, "x_B"] \\ & X & \end{tikzcd} :\ \exists j\in J_{(A,B)} :\ \begin{tikzcd} A \arrow[r, tail, "a_j"] \arrow[dr, tail, bend right, "x_A"'] & M_j \arrow[d, tail, dashed, "\exists!\, m_j"] & B \arrow[l, tail, "b_j"'] \arrow[dl, tail, bend left, "x_B"] \\ & X & \end{tikzcd}$$

$$\wedge\ \forall \begin{tikzcd} A \arrow[dr, tail, "y_A"] \arrow[ddr, tail, bend right, "x_A"'] & & B \arrow[dl, tail, "y_B"'] \arrow[ddl, tail, bend left, "x_B"] \\ & Y \arrow[d, tail, "y_X"] & \\ & X & \end{tikzcd} :\ \exists!\ M_j \overset{m}{\rightarrowtail} Y :\ \begin{tikzcd} A \arrow[r, tail, "a_j"] \arrow[dr, tail, "y_A"] \arrow[ddr, tail, bend right, "x_A"'] & M_j \arrow[d, tail, "m"] & B \arrow[l, tail, "b_j"'] \arrow[dl, tail, "y_B"'] \arrow[ddl, tail, bend left, "x_B"] \\ & Y \arrow[d, tail, "y_X"] & \\ & X & \end{tikzcd}$$

DEFINITION: CLEAVAGE FOR MULTI-SUMS:

$$\forall\ (A,B)\in\mathbb{D}_0\times\mathbb{D}_0 :\ ms(A,B) = \left\{ A \overset{a_j}{\rightarrowtail} M_j \overset{b_j}{\leftarrowtail} B \right\}_{J^{\circ}_{(A,B)}}$$

15

$\mathbb{D}_0$ HAS MULTI-SUMS:

$\forall (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 : \exists \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{j \in J_{(A,B)}} : \forall$

$$\begin{tikzcd} A \arrow[dr, "x_A"'] & & B \arrow[dl, "x_B"] \\ & X & \end{tikzcd}$$

$: \exists j \in J_{(A,B)} :$

$$\begin{tikzcd} A \arrow[r, "a_j"] \arrow[dr, bend right, "x_A"'] & M_j \arrow[d, dotted, "\exists!\, m_j"] & B \arrow[l, "b_j"'] \arrow[dl, bend left, "x_B"] \\ & X & \end{tikzcd}$$

$\wedge \; \forall$

$$\begin{tikzcd} A \arrow[dr, "y_A"] \arrow[ddr, bend right, "x_A"'] & & B \arrow[dl, "y_B"'] \arrow[ddl, bend left, "x_B"] \\ & Y \arrow[d, "y_X"] & \\ & X & \end{tikzcd}$$

$: \exists! \; M_j \xrightarrow{m} Y :$

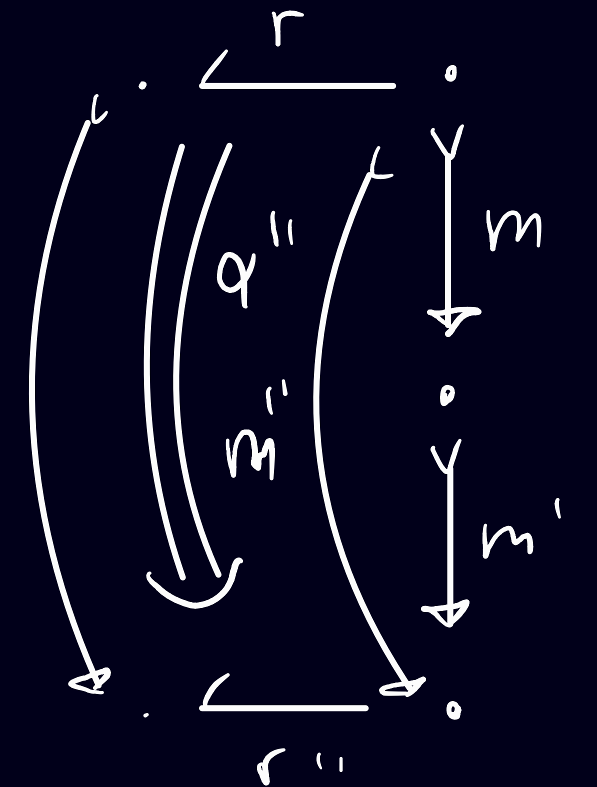
$$\begin{tikzcd} A \arrow[r, "a_j"] \arrow[dr, "y_A"'] \arrow[ddr, bend right, "x_A"'] & M_j \arrow[d, "m"] & B \arrow[l, "b_j"'] \arrow[dl, "y_B"] \arrow[ddl, bend left, "x_B"] \\ & Y \arrow[d, "y_X"] & \\ & X & \end{tikzcd}$$

DEFINITION: CLEAVAGE FOR MULTI-SUMS:

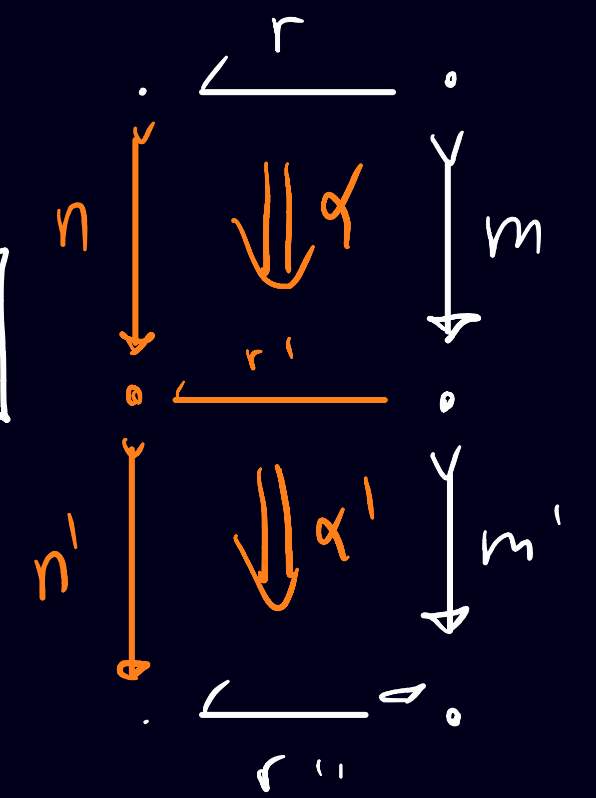
$\forall (A,B) \in \mathbb{D}_0 \times \mathbb{D}_0 : ms(A,B) = \left\{ A \xrightarrow{a_j} M_j \xleftarrow{b_j} B \right\}_{j \in J^{\circ}_{(A,B)}}$

$S : \mathbb{D}_1 \to \mathbb{D}_0$ IS A MULTI-OPFIBRATION

$\forall$



$: \exists$

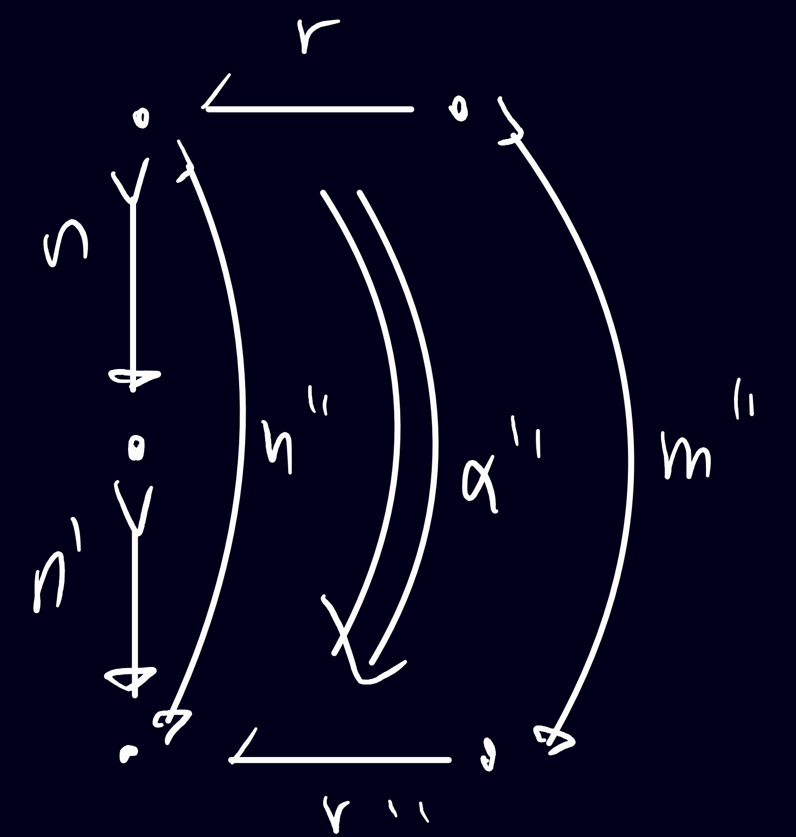


$: n' \circ n = n''$

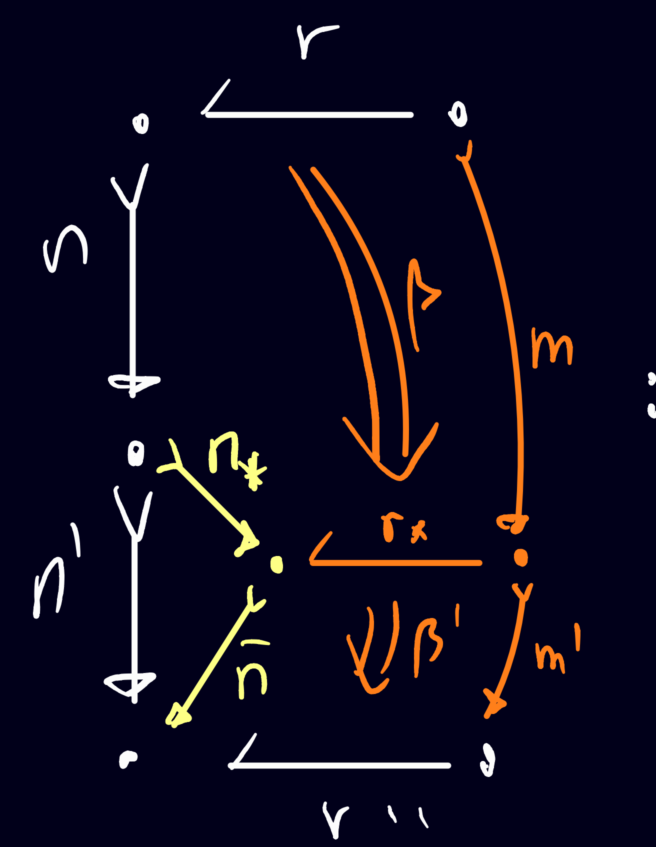
$\wedge \; \alpha' \circ_v \alpha = \alpha''$

$T : \mathbb{D}_1 \to \mathbb{D}_0$ IS A RESIDUAL MULTI-OPFIBRATION

$\forall$



$: \exists$



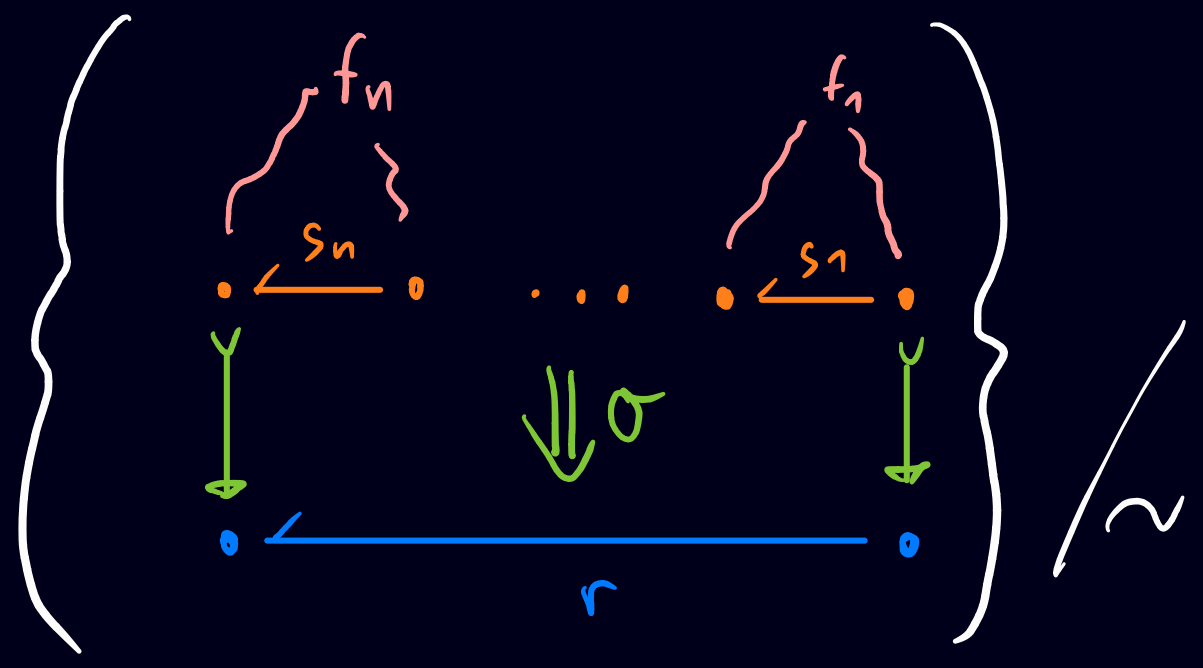
$: \bar{n} \circ n_x = n'$

$\wedge \; m' \circ m = m''$

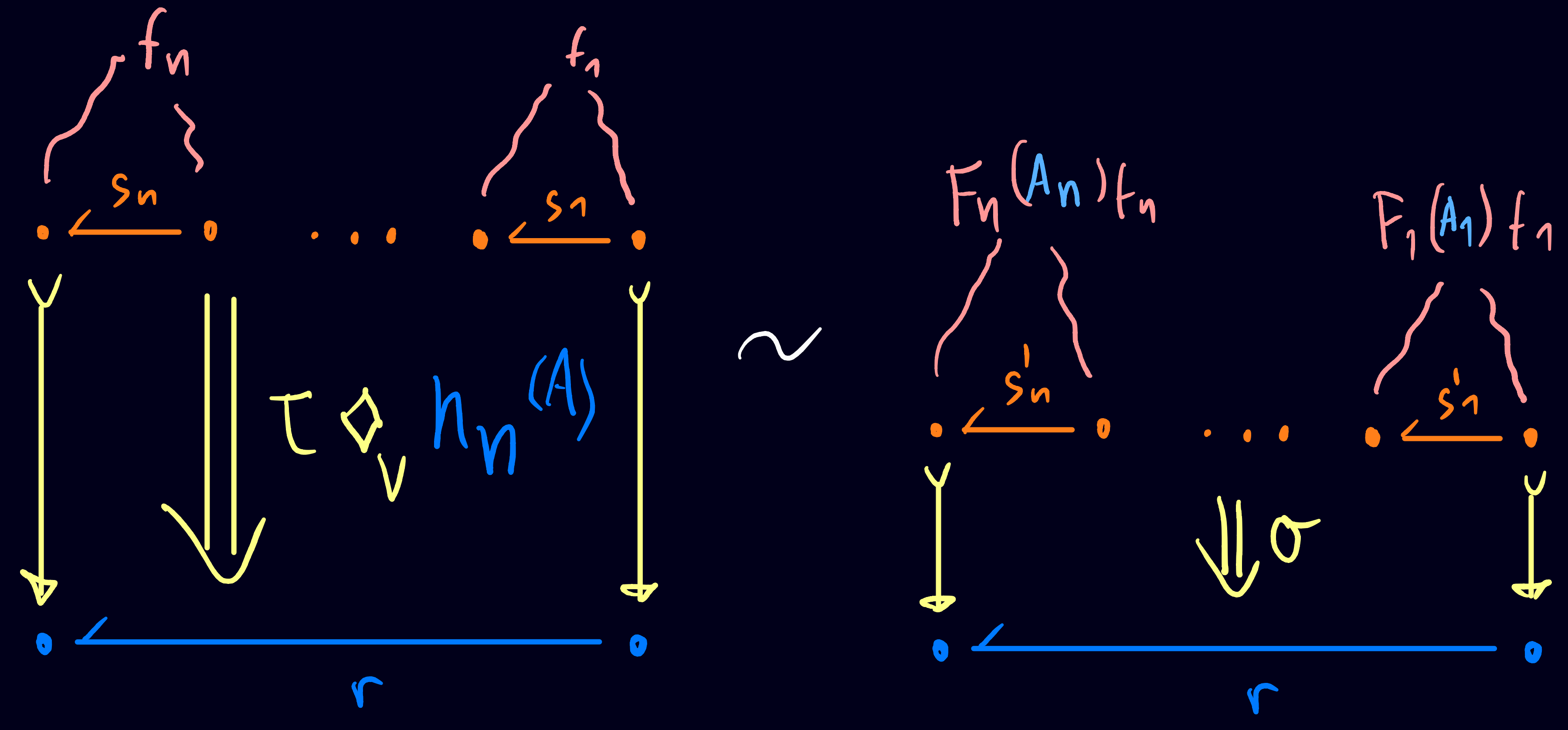
$\wedge \; \beta' \circ_v \beta = \beta''$

16 CONVOLUTION PRODUCTS REVISITED

RECAP: $(F_n * \cdots * F_1)(r) := \int^{S=(s_n,\ldots,s_1)\in\mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) =$

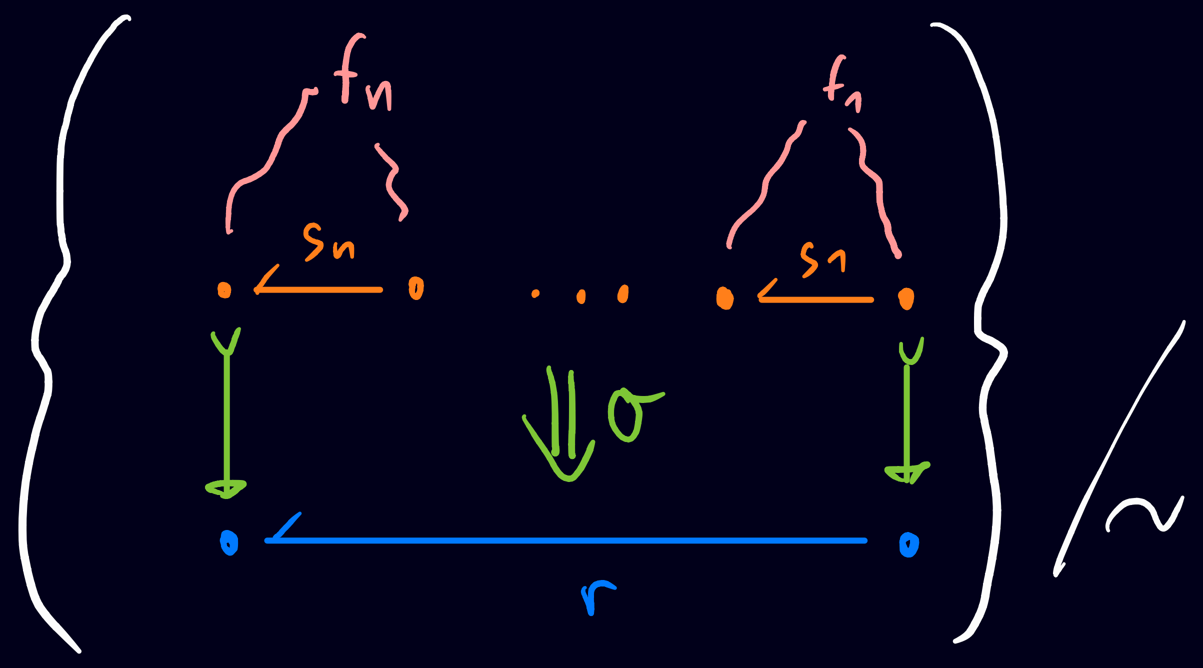


$\forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,\ h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:

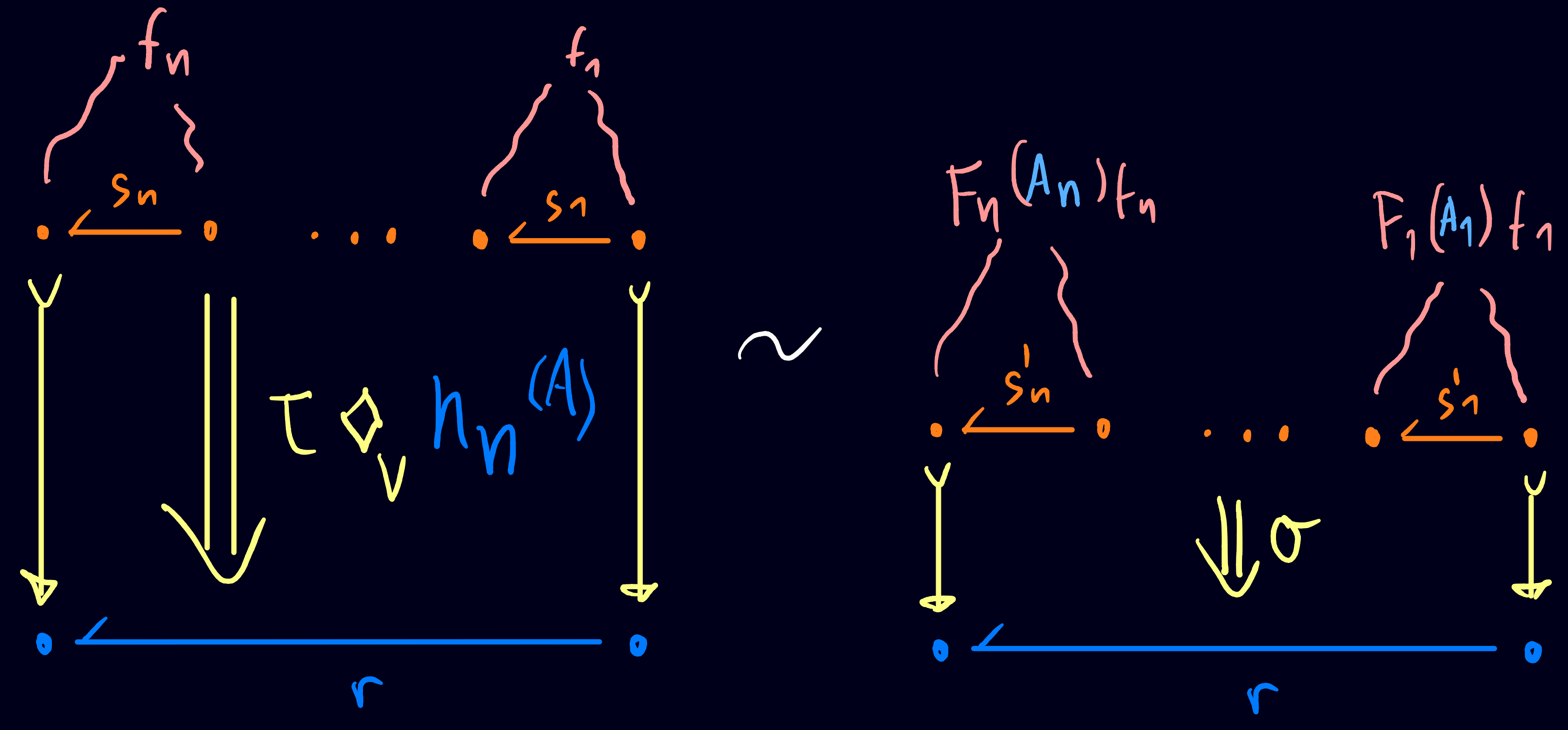


16 CONVOLUTION PRODUCTS REVISITED

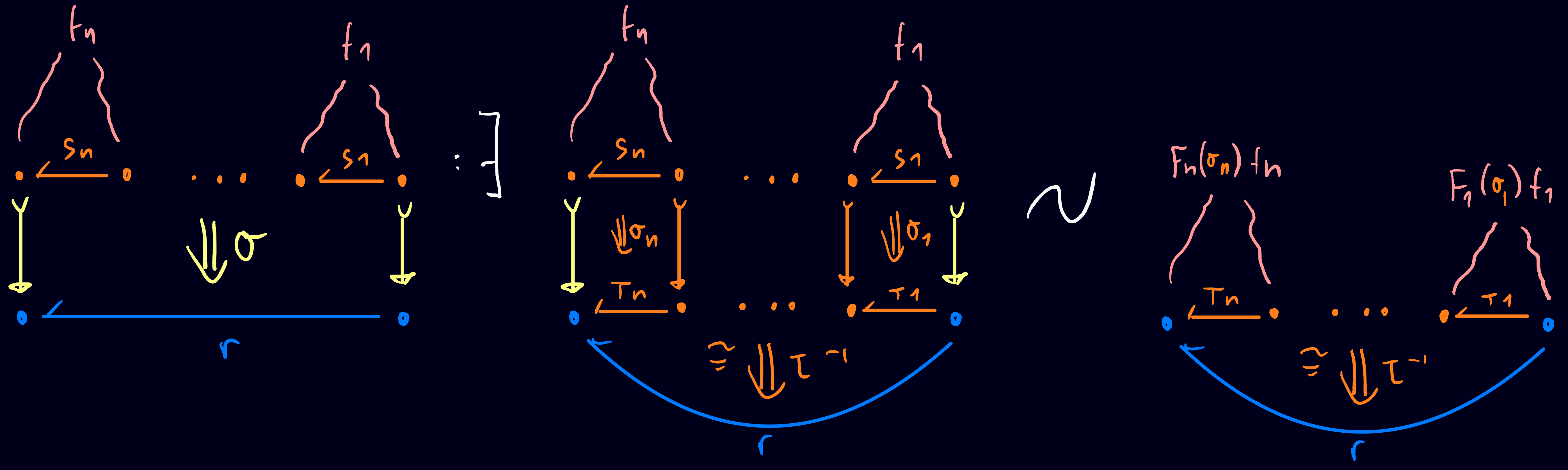
RECAP: $(F_n * \cdots * F_1)(r) := \int^{S=(S_n,\dots,S_1)\in \mathbb{D}_n} \mathbb{D}_1(h_n(S), r) \times \mathbb{F}_n(S) =$



$\forall\, S \xrightarrow{A} S' \in \mathbb{D}_n,$
$h_n(S') \xrightarrow{\tau} r \in \mathbb{D}_1$:

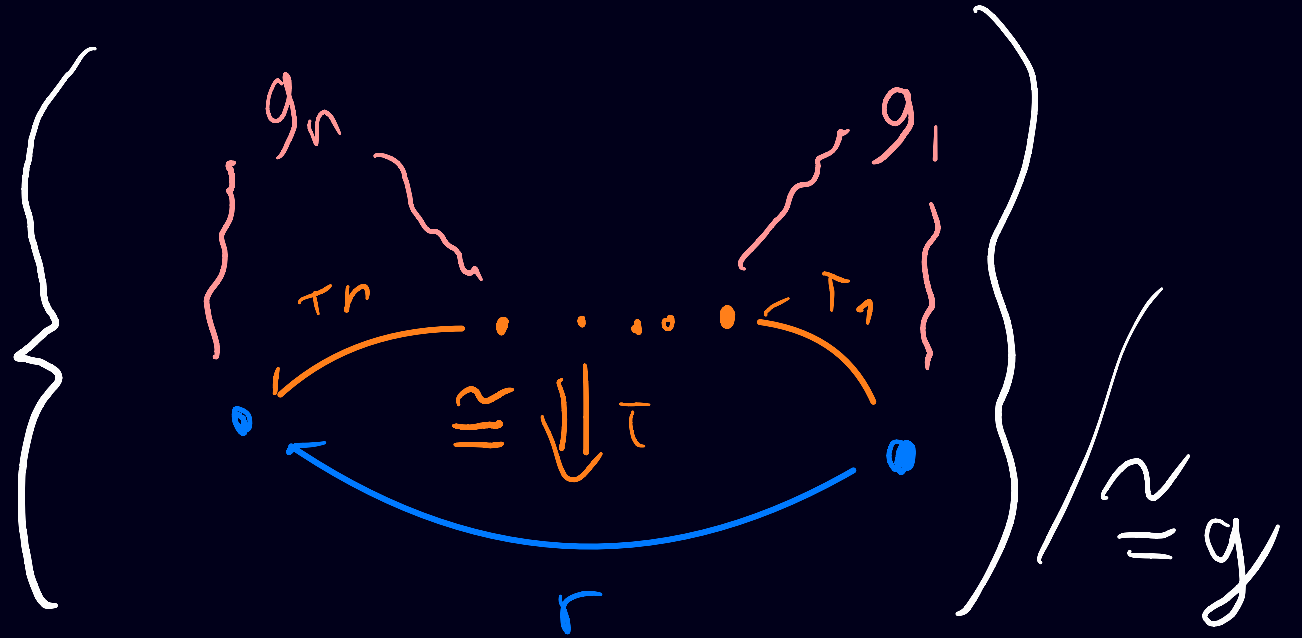


<u>NOW</u>: $h_n : \mathbb{D}_n \to \mathbb{D}_1$ IS A "GLOBULAR" STREET OPFIBRATION $\Rightarrow \forall$

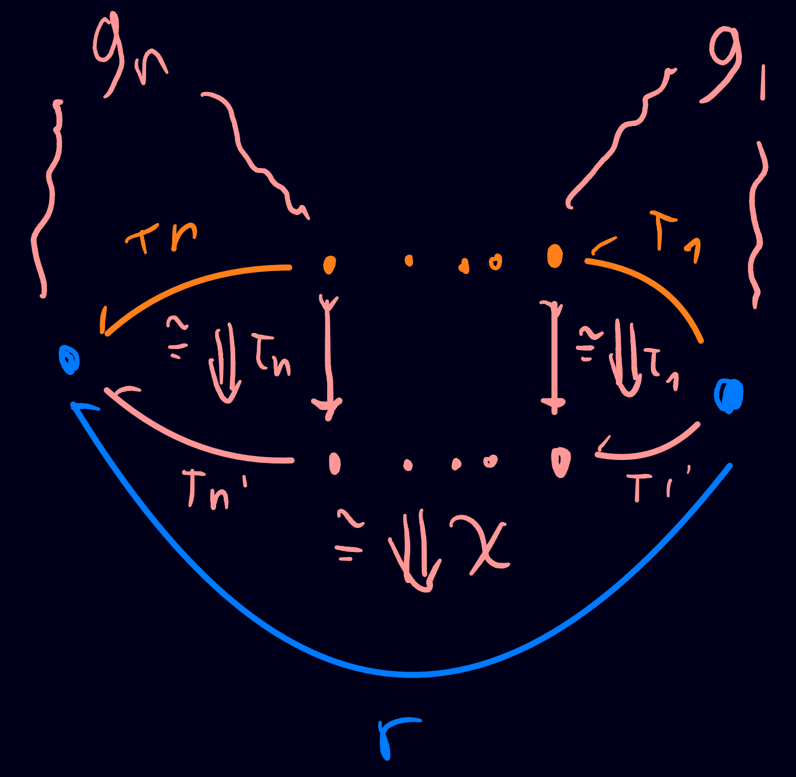


17

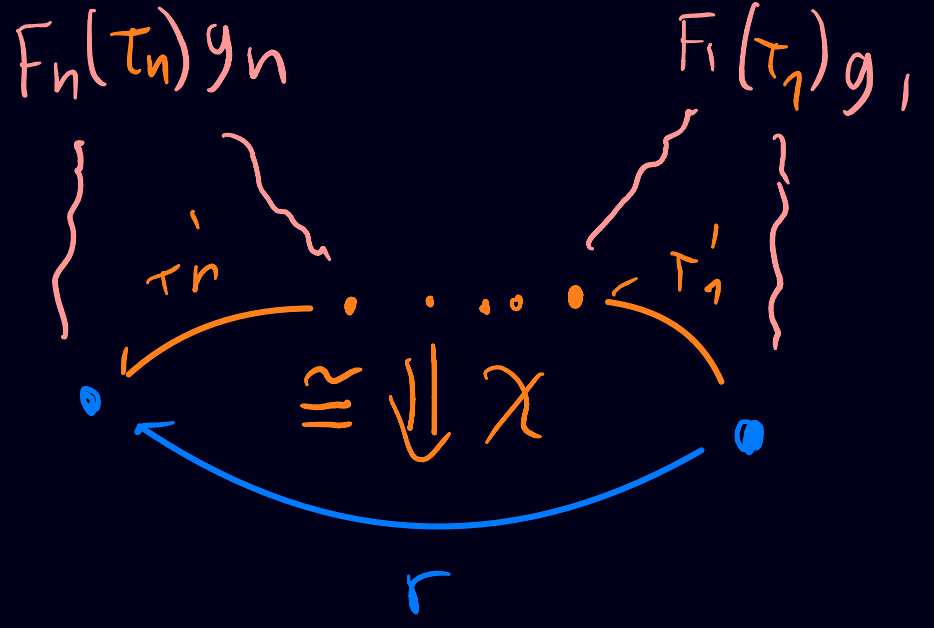
↳ $(F_n * \dots * F_1)(r) \cong$



WHERE

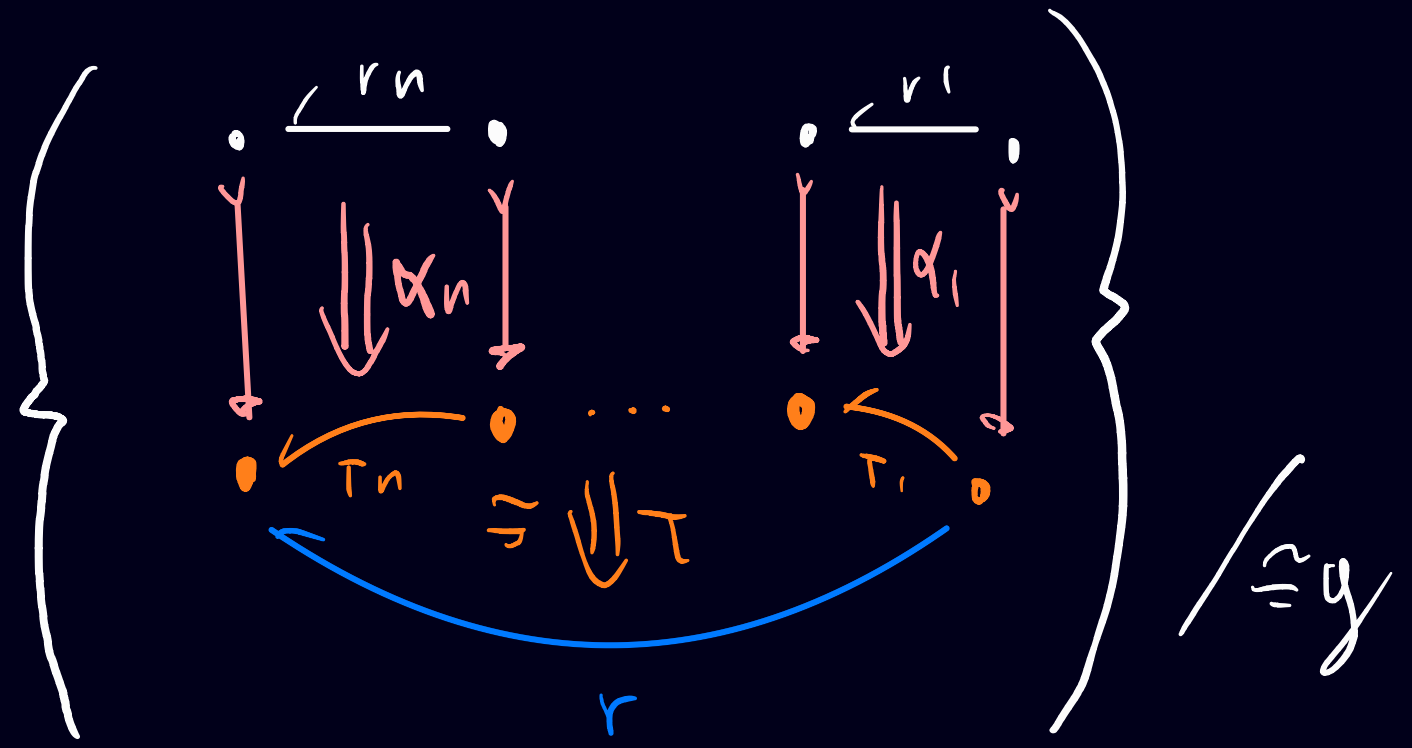


$\cong_g$



EXAMPLE: FOR $\hat{\Delta}_{r_j} := \mathbb{D}_1(r_j, -)$ $(j=1,\dots,n)$

$(\hat{\Delta}_{r_n} * \dots * \hat{\Delta}_{r_1})(r) \cong$



✓

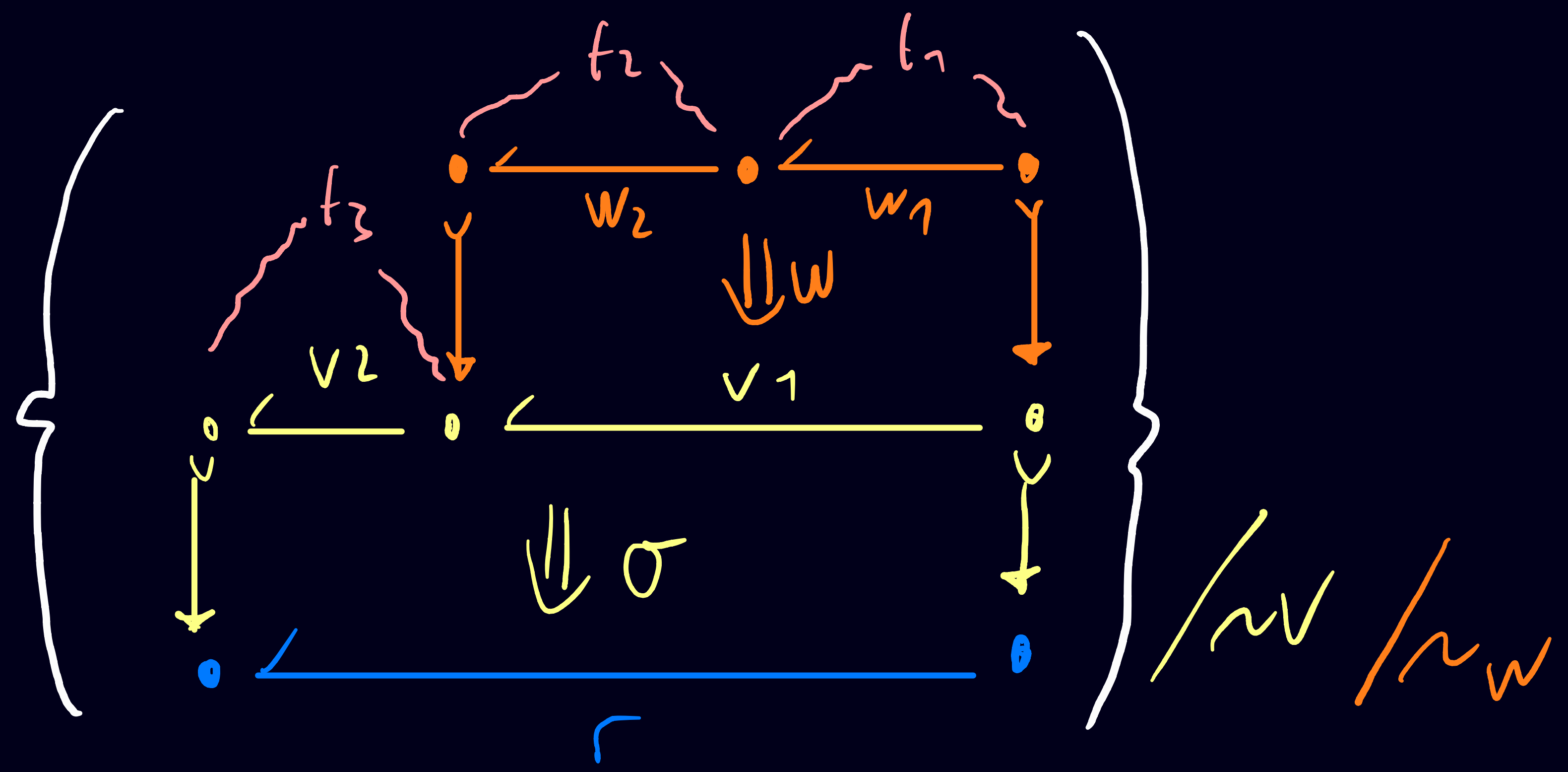
18 KEY RESULT: WEAK ASSOCIATIVITY OF $*$

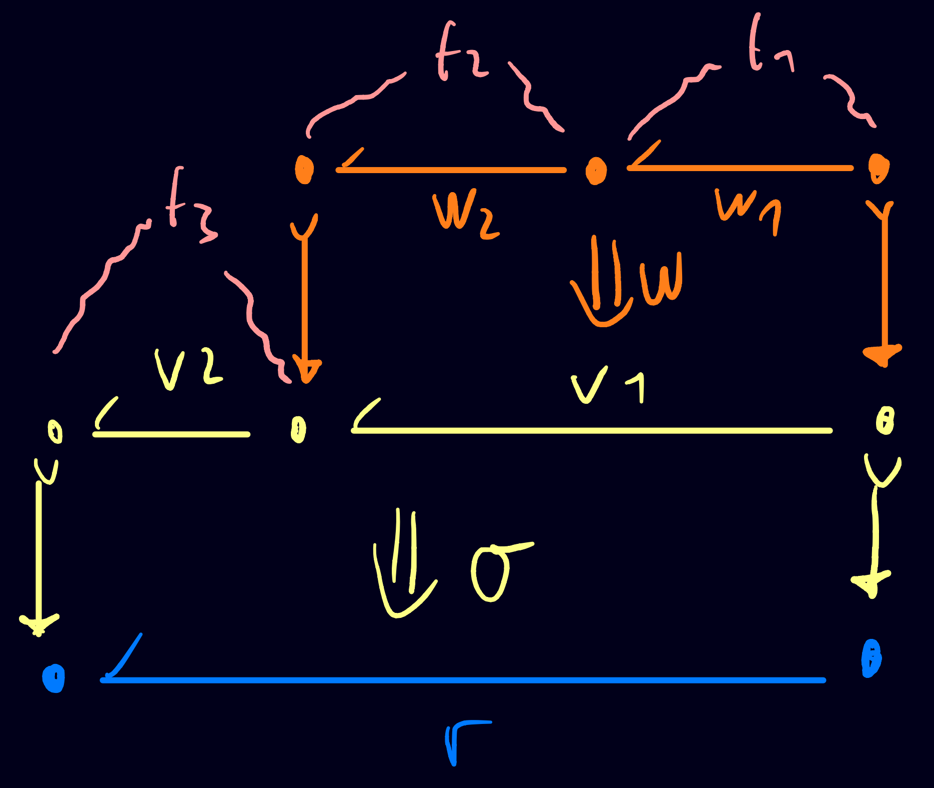
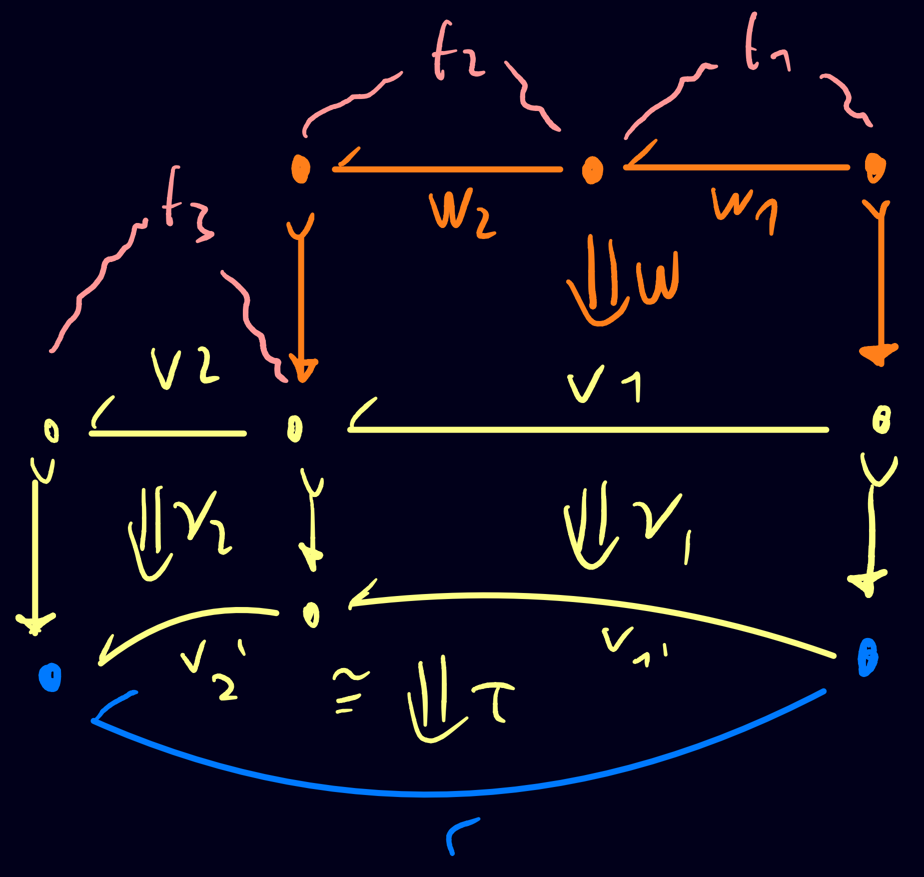
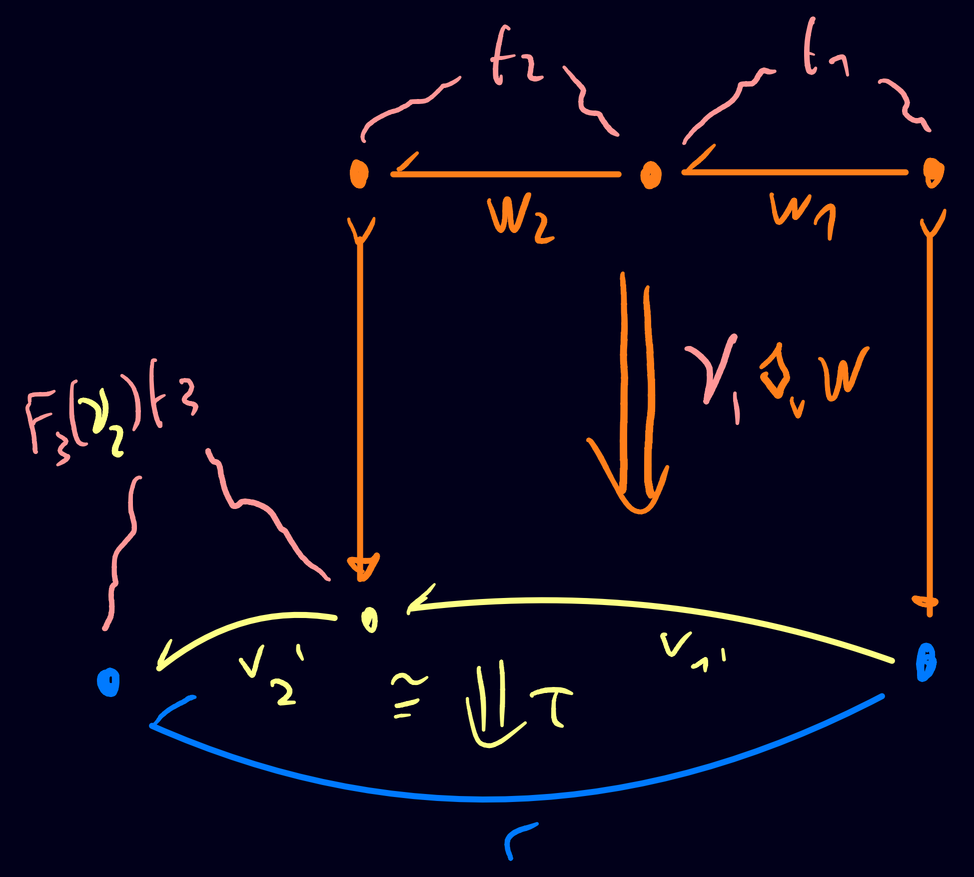
$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{\mathbf{Set}}\,,\; r \in \mathbb{D}_1 : \quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

18 KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$$\forall F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{\mathbf{Set}}\,,\ r \in \mathbb{D}_1:\ F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

PROOF (SKETCH):

$F_3 * (F_2 * F_1)(r) = $ 

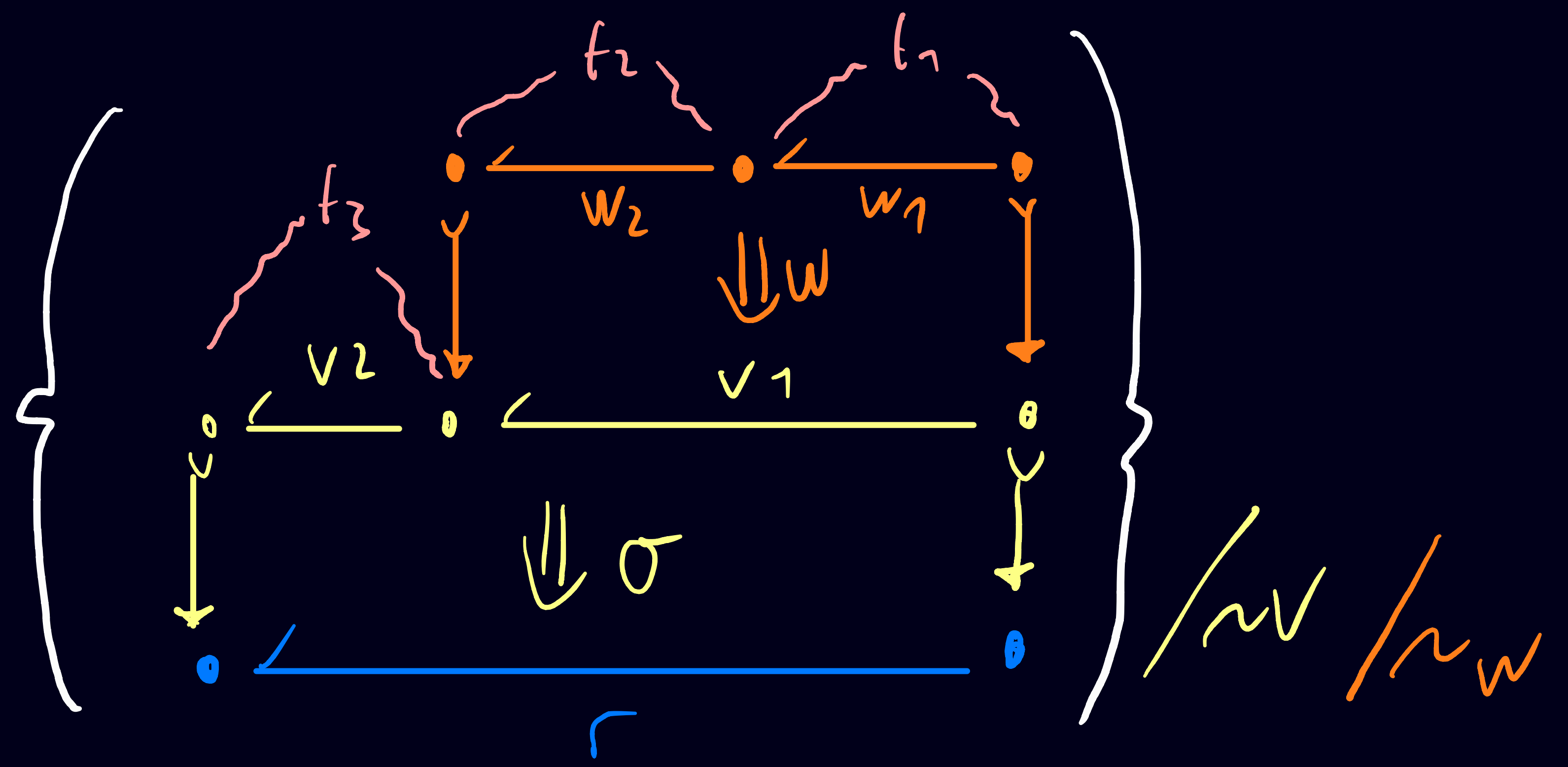
$\forall$  $: \exists$  $\sim_V$ 

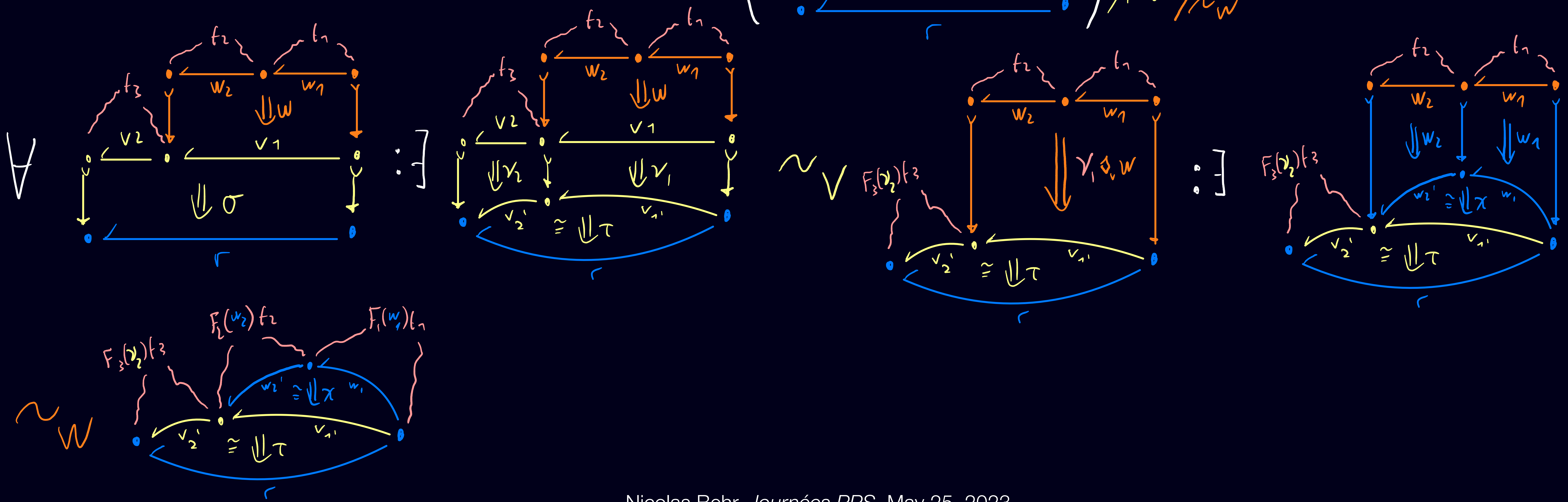
18 KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{\mathbf{Set}}\,,\quad r \in \mathbb{D}_1 : \quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

PROOF (SKETCH):

$$F_3 * (F_2 * F_1)(r) =$$



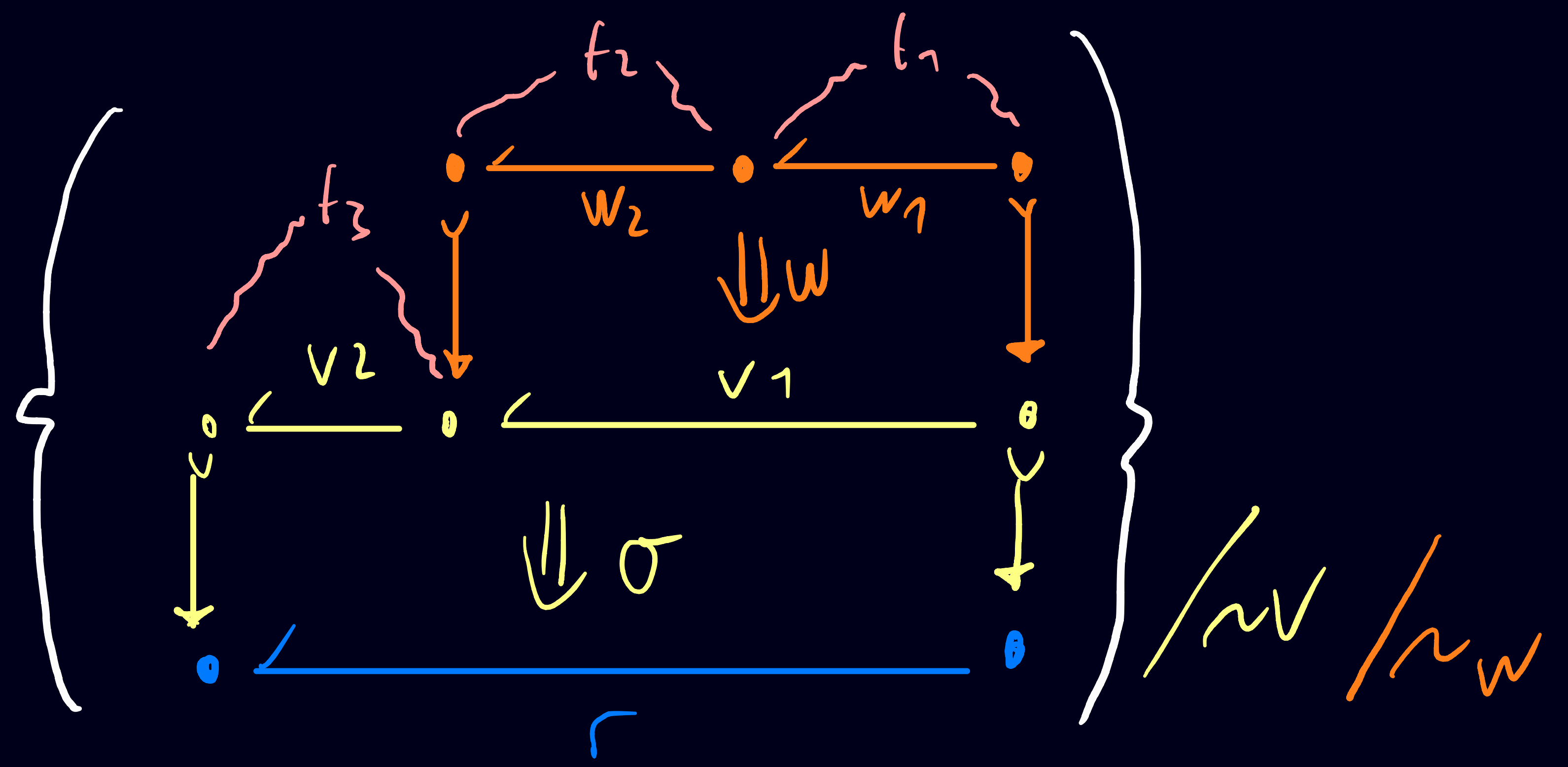


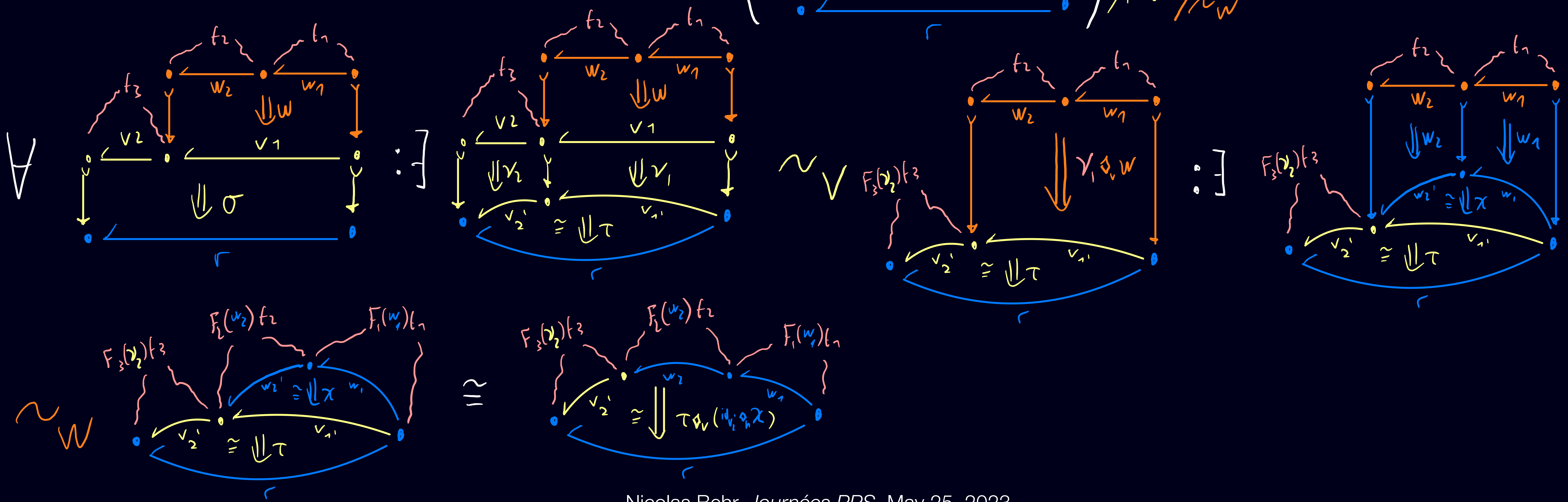
18 KEY RESULT: WEAK ASSOCIATIVITY OF $*$

$$\forall\, F_3, F_2, F_1 : \mathbb{D}_1 \longrightarrow \underline{\mathbf{Set}}\,,\quad r \in \mathbb{D}_1 :\quad F_3 * (F_2 * F_1)(r) \cong (F_3 * F_2 * F_1)(r) \cong (F_3 * F_2) * F_1(r)$$

PROOF (SKETCH):

$$F_3 * (F_2 * F_1)(r) =$$





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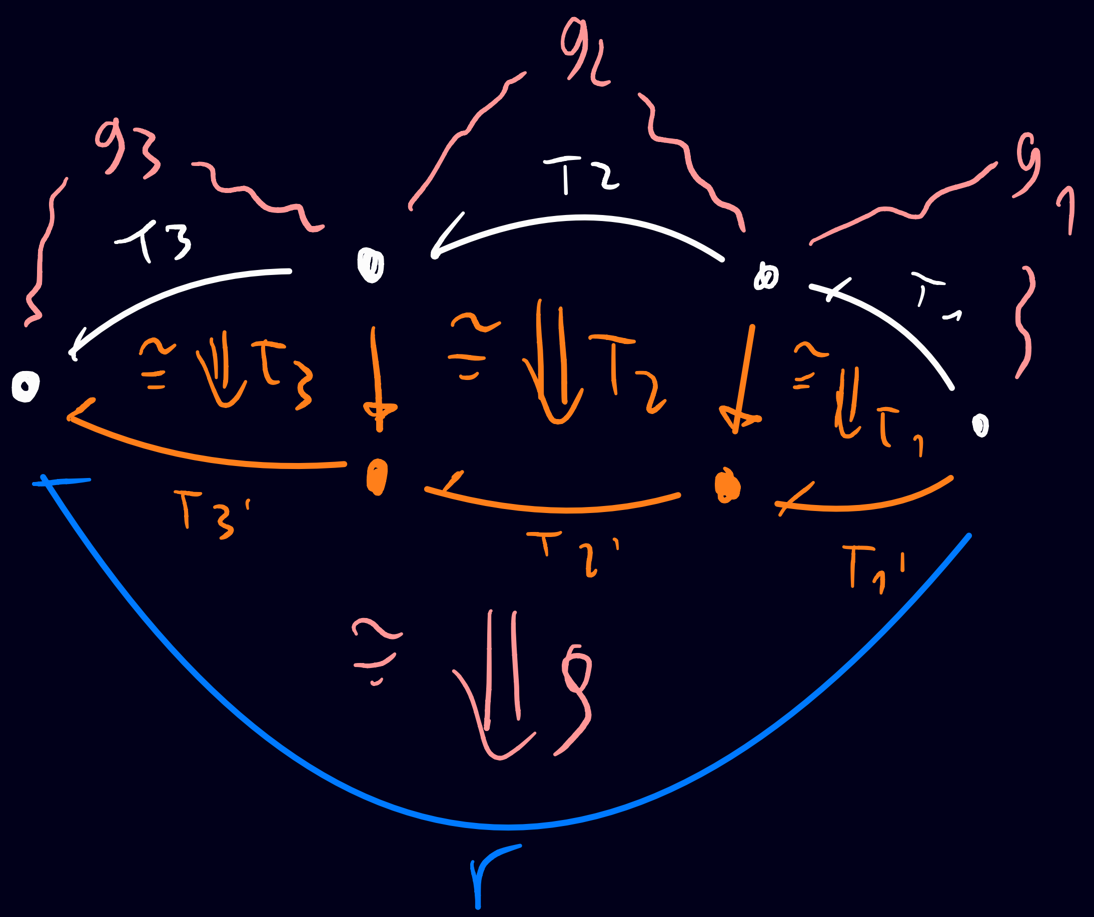
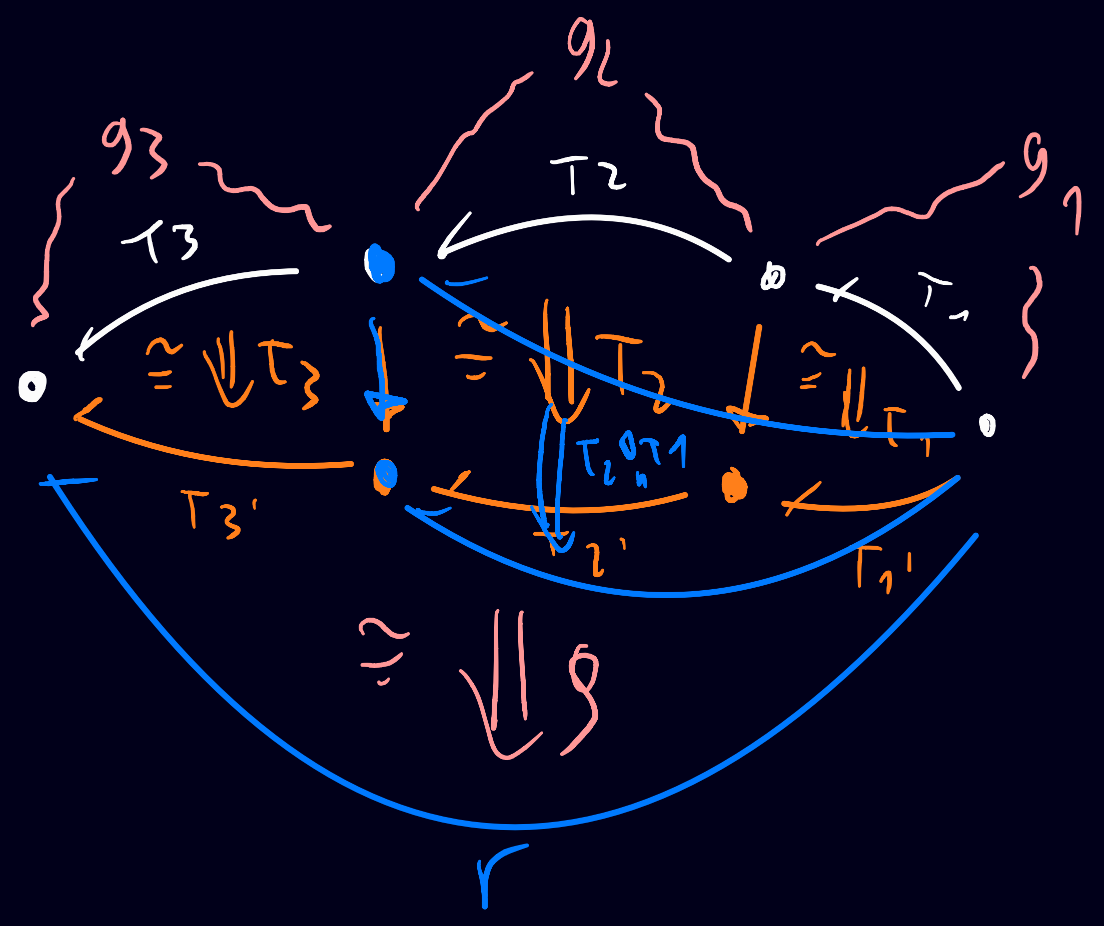
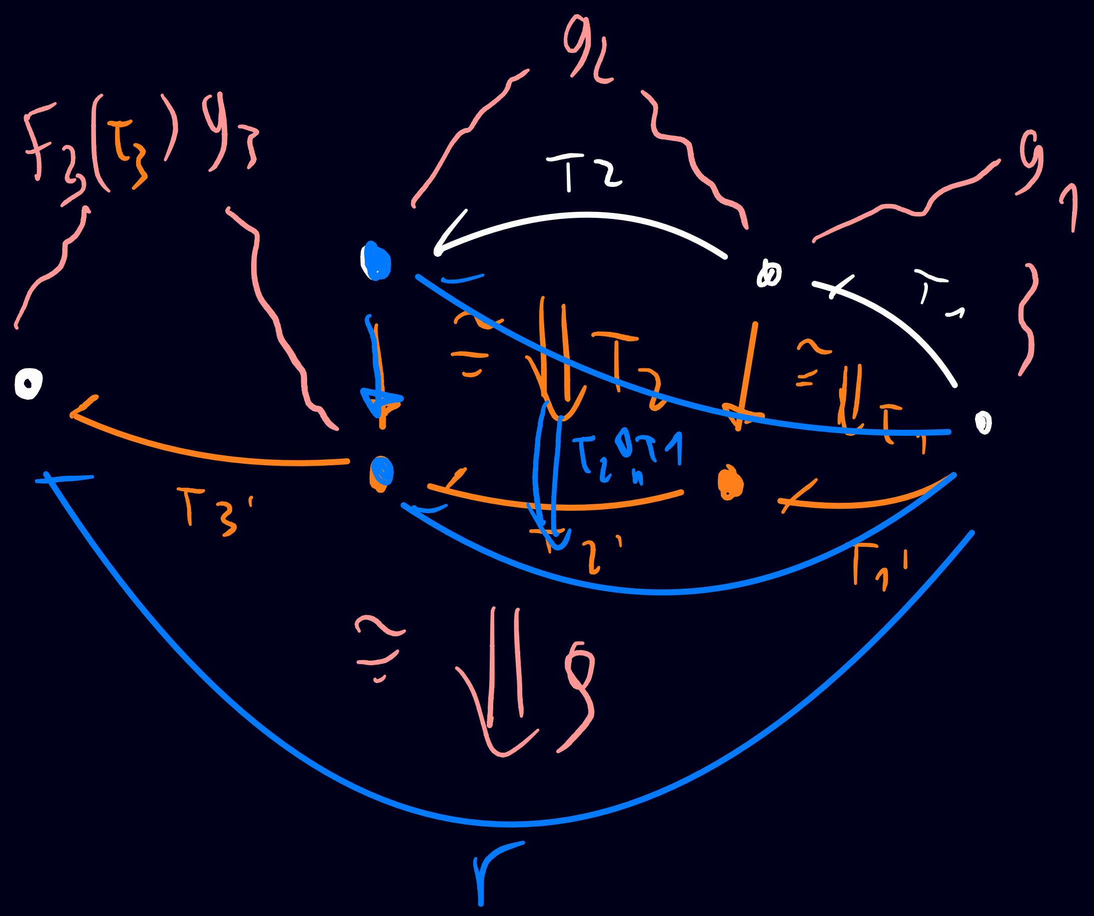

g3
g2
g1
T3
T2
T1
≃ ⇓ T3
≃ ⇓ T2
≃ ⇓ T1
T3'
T2'
T1'
≃ ⇓ g
r

: ∃

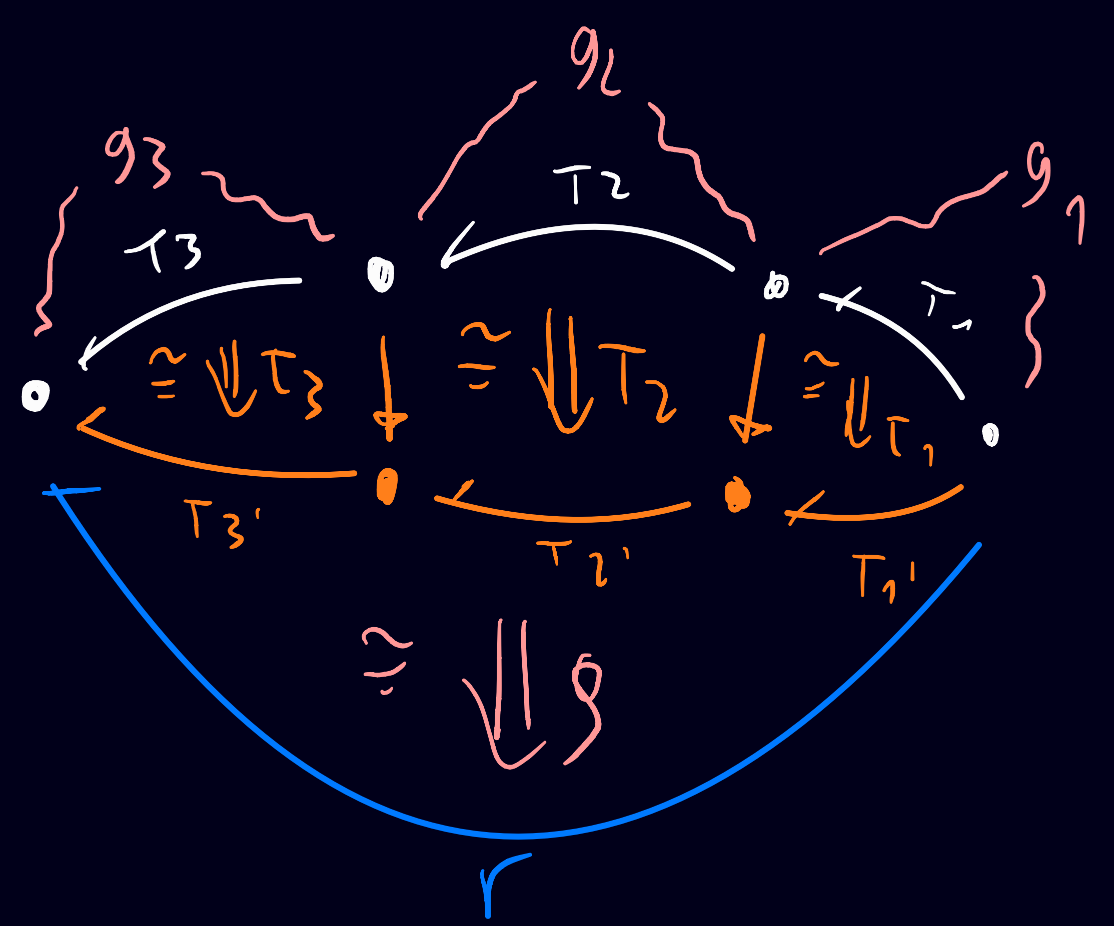

g3
g2
g1
T3
T2
T1
≃ ⇓ T3
≃ ⇓ T2
≃ ⇓ T1
T2 ⊗ T1
T3'
T2'
T1'
≃ ⇓ g
r

∼ √

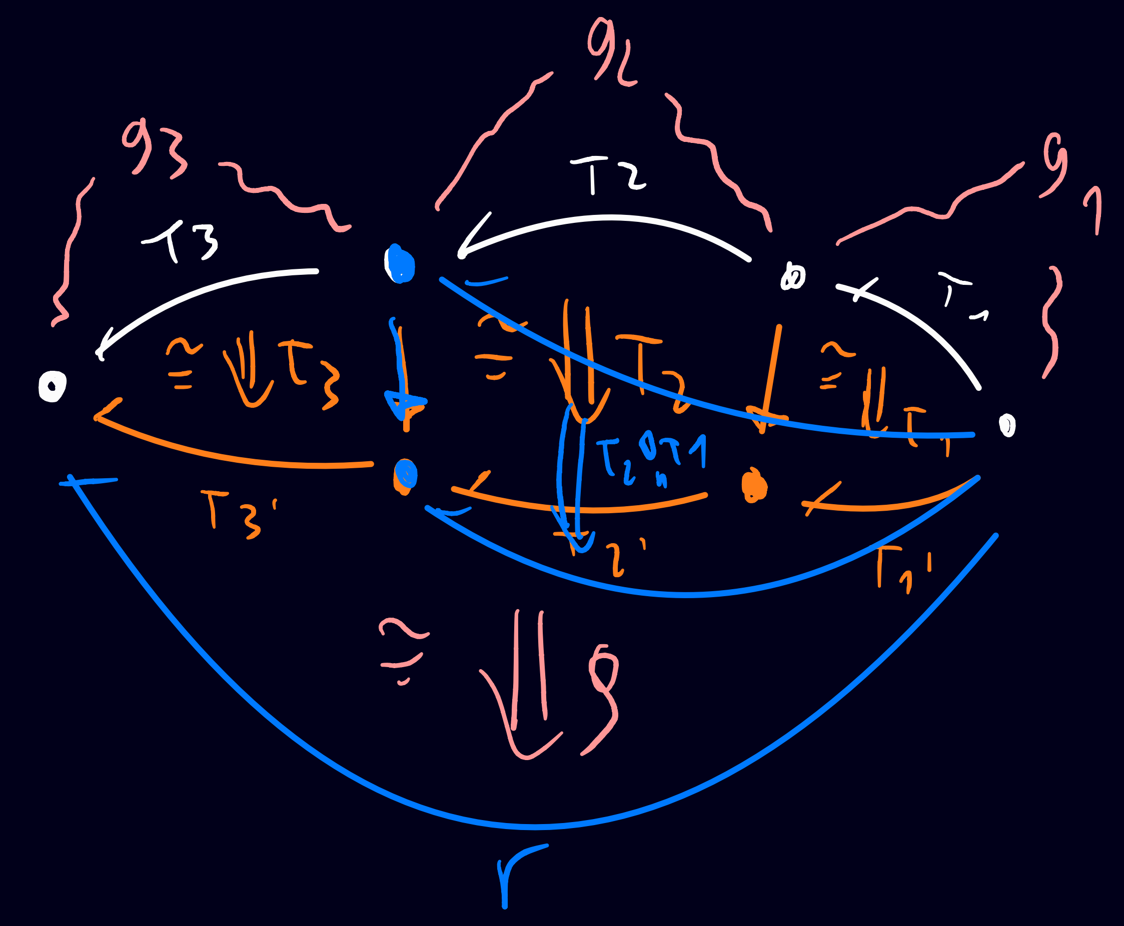

F3(T3) g3
g2
g1
T2
T1
≃ ⇓ T2
≃ ⇓ T1
T2 ⊗ T1
T3'
T2'
T1'
≃ ⇓ g
r

19

$\cdot\ \forall$



$: \exists$



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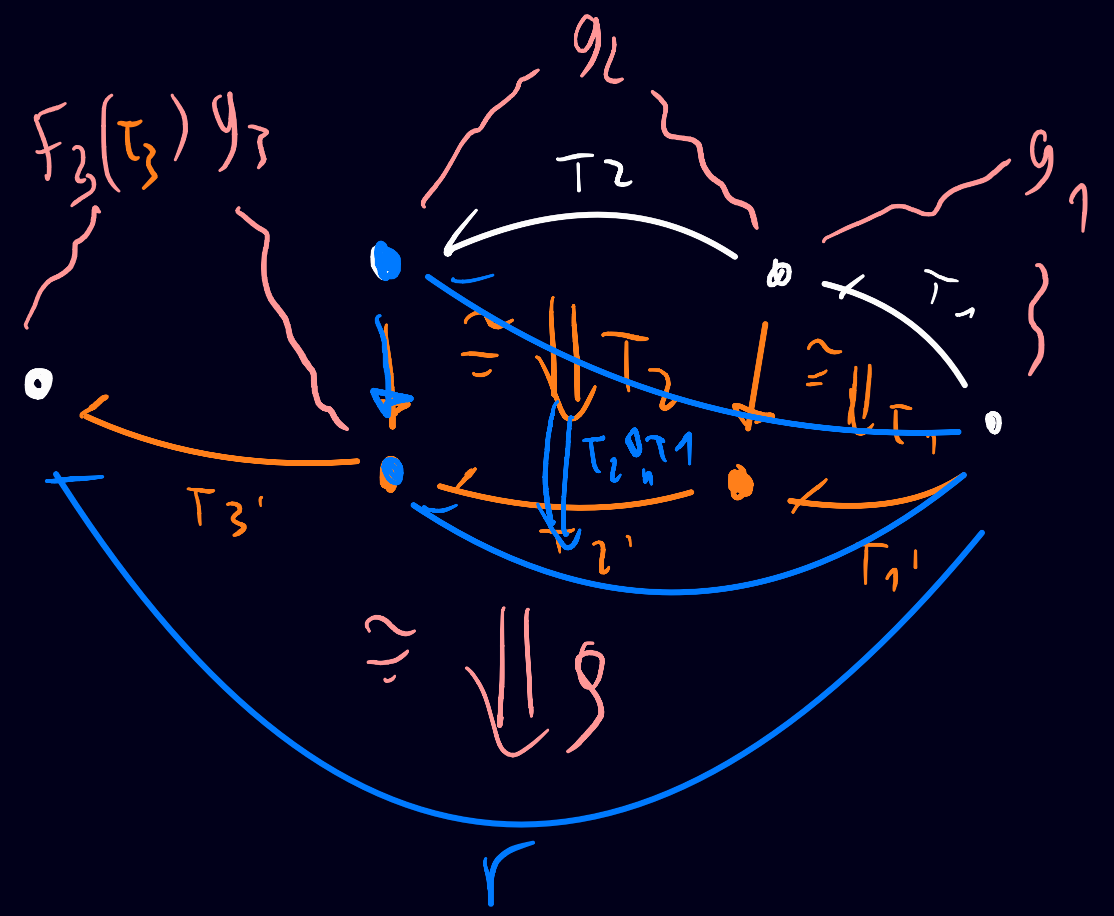

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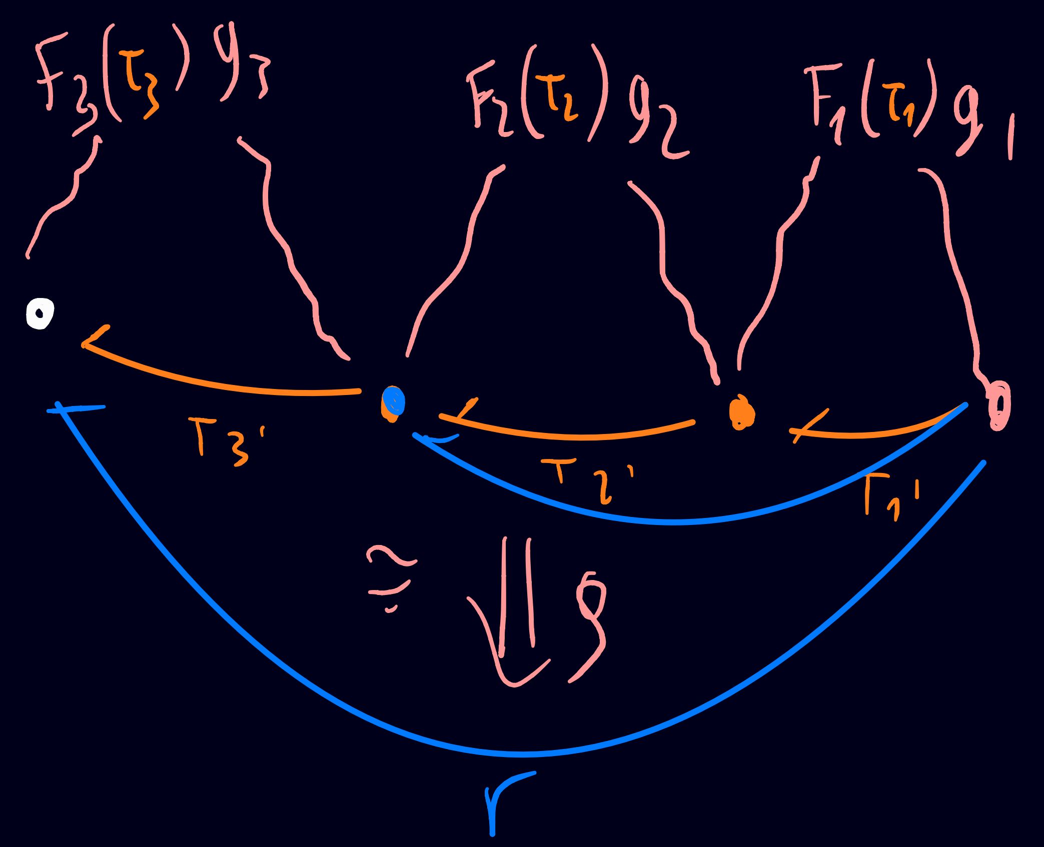
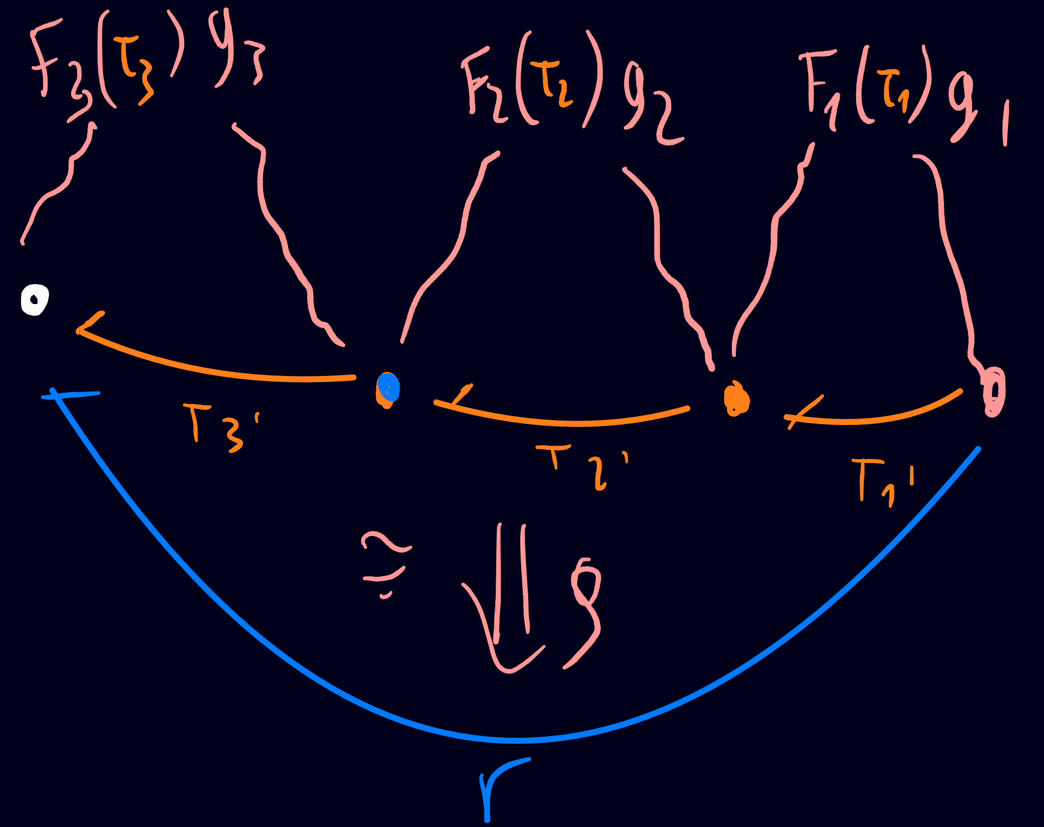
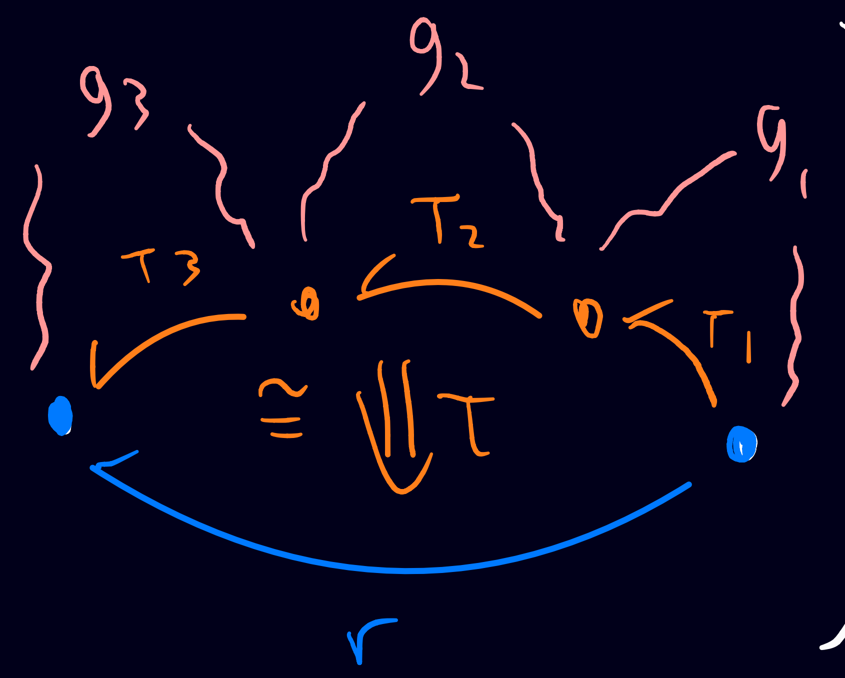


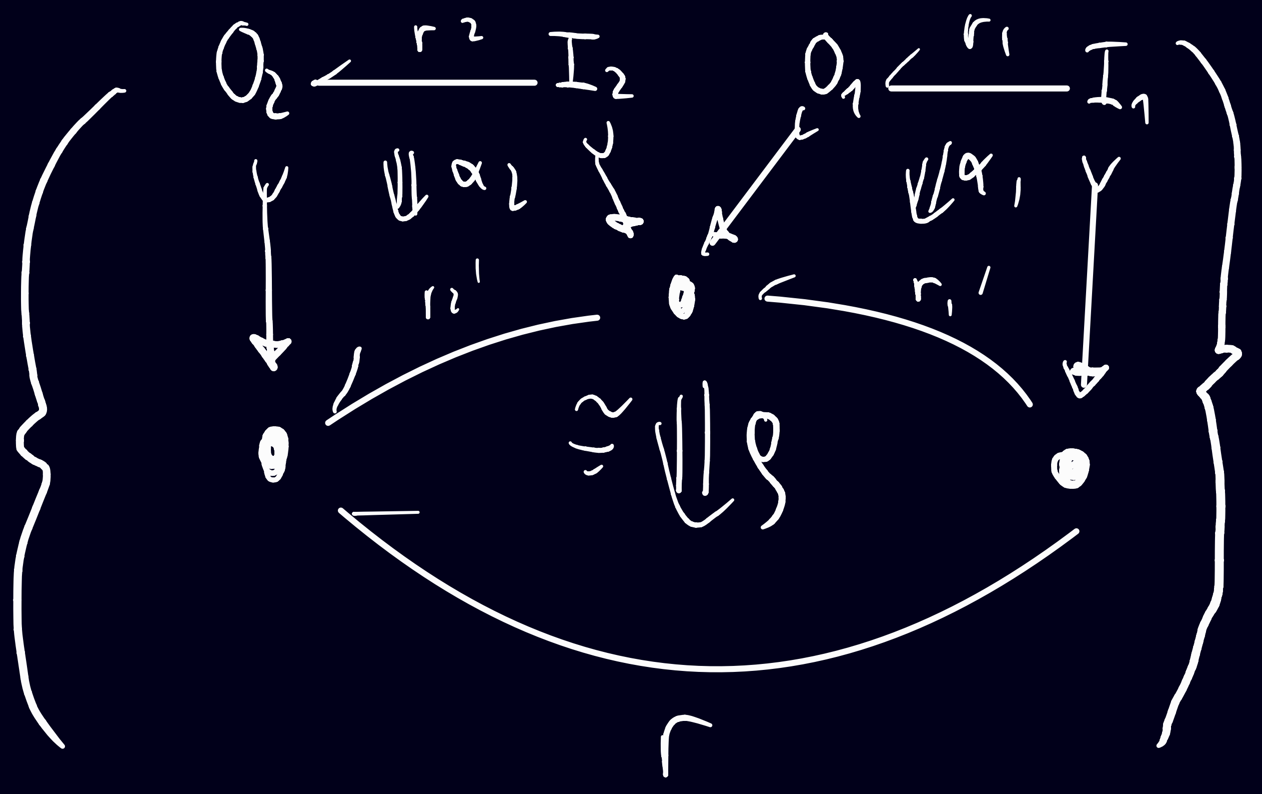
$\cong$

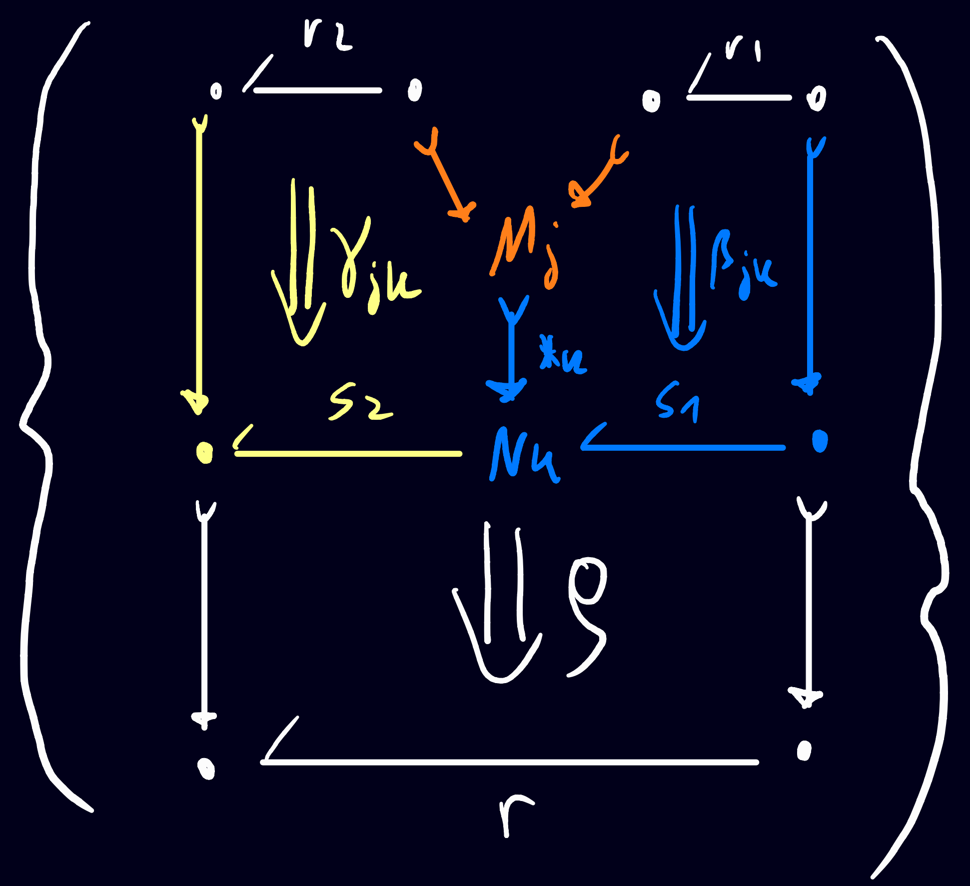


$\hookrightarrow\ F_3 * (F_2 * F_1)(r) \cong \Big\{$  $\Big\} / {\cong_g} = (F_3 * F_2 * F_3)(r)$

$\square$

20 FINAL INGREDIENT: CATEGORIFICATION OF RULE ALGEBRA

CLAIM: $(\hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong \Bigg\{$  $\Bigg\} \Big/ \cong_y$

$$\cong \coprod_{\substack{(O_2 \rightarrowtail M_j \leftarrowtail I_1) \\ \in \mathsf{ms}(O_2, I_1)}} \ \coprod_{\beta_{jk} \in T^*(O_1 \rightarrowtail M_j)} \ \coprod_{\gamma_{jk} \in S^*(I_2 \rightarrowtail M_j \xrightarrow{*_{k}} N_k)}$$ $\Bigg\{$  $\Bigg\}$

$$=: \hat{\Delta}_{r_2 \circ r_1}(r)$$

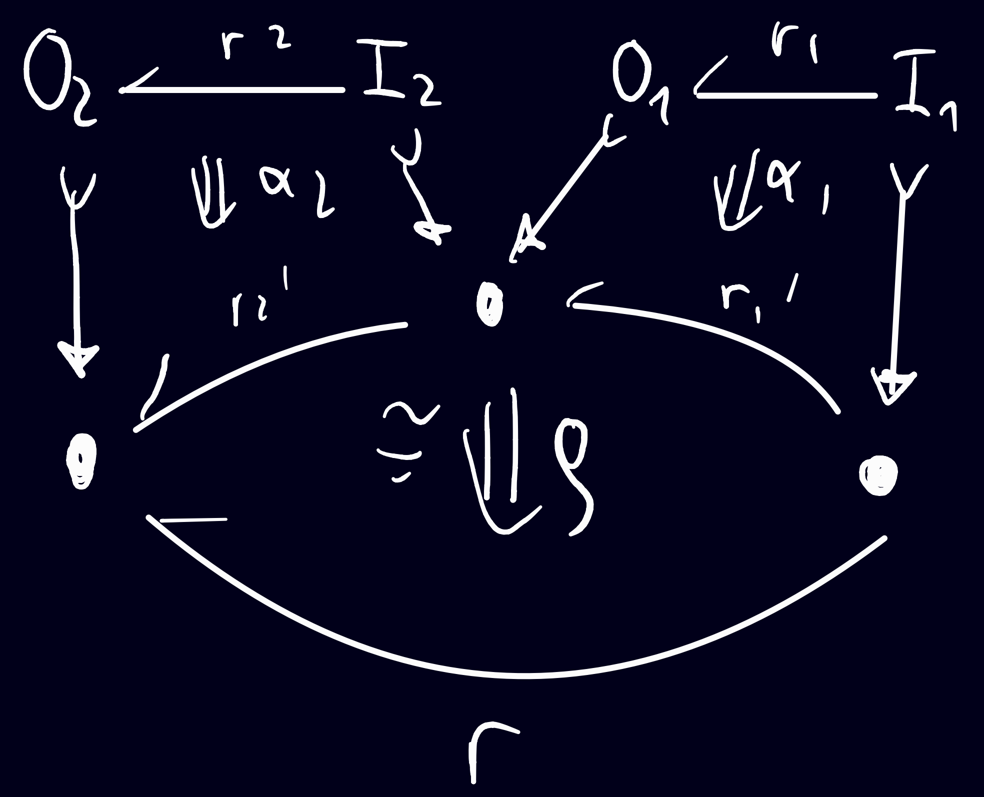
(21) Proof (sketch): Assuming chosen cleavages:

$$\forall \quad \begin{tikzcd}
O_2 \arrow[d] \arrow[r, phantom, "\Downarrow\alpha_2"] & I_2 \arrow[l, "r_2"'] \arrow[dr] & & O_1 \arrow[dl] \arrow[r, phantom, "\Downarrow\alpha_1"] & I_1 \arrow[l, "r_1"'] \arrow[d] \\
\bullet & & \bullet \arrow[ll, "r_2'"'] \arrow[d, phantom, "\cong\Downarrow\rho"] & & \bullet \arrow[ll, "r_1'"] \arrow[llll, bend left, "r"] \\
 & & {} & &
\end{tikzcd} \quad : \quad \exists!$$

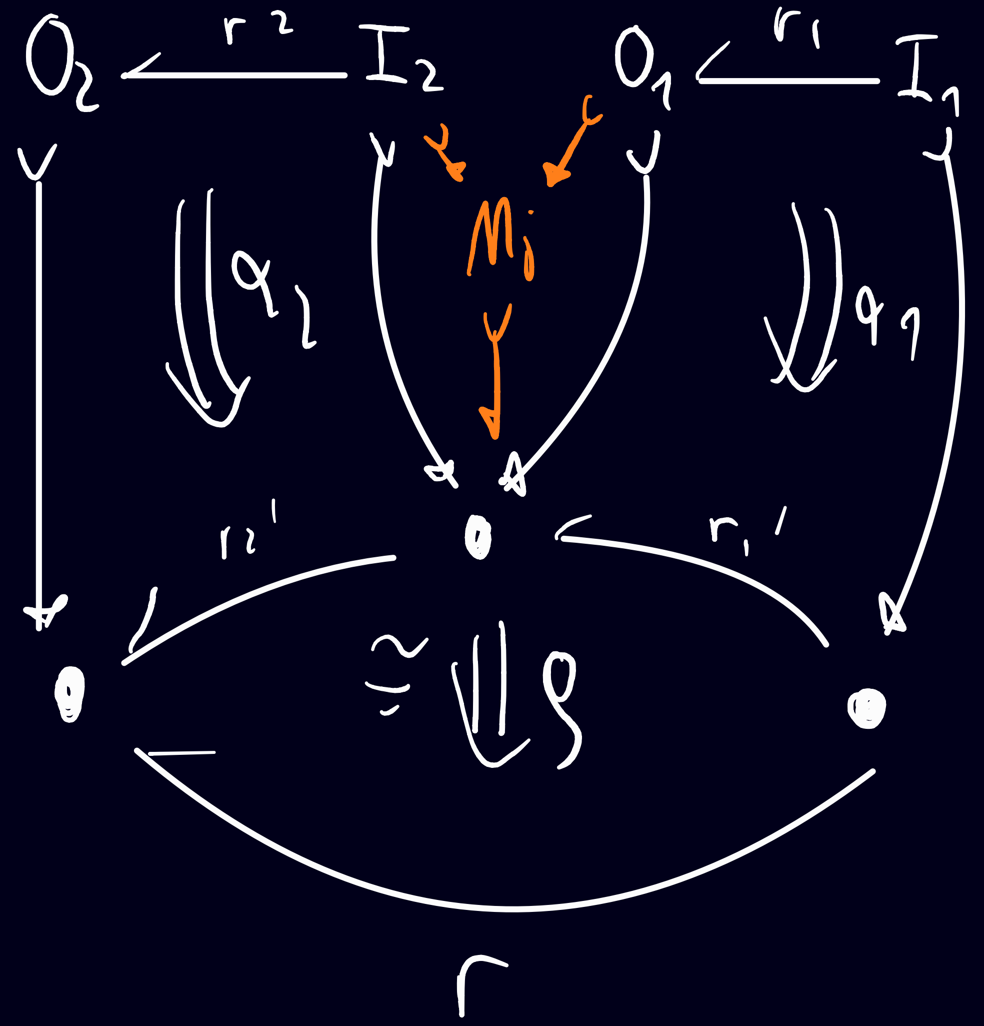
$$\begin{tikzcd}
O_2 \arrow[dd] \arrow[r, phantom, "\Downarrow\alpha_2"] & I_2 \arrow[l, "r_2"'] \arrow[dr] \arrow[ddr, bend right] & & O_1 \arrow[dl] \arrow[ddl, bend left] \arrow[r, phantom, "\Downarrow\alpha_1"] & I_1 \arrow[l, "r_1"'] \arrow[dd] \\
 & & M_i \arrow[d] & & \\
\bullet & & \bullet \arrow[ll, "r_2'"'] \arrow[d, phantom, "\cong\Downarrow\rho"] & & \bullet \arrow[ll, "r_1'"] \arrow[llll, bend left, "r"] \\
 & & {} & &
\end{tikzcd}$$

⑵① PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:

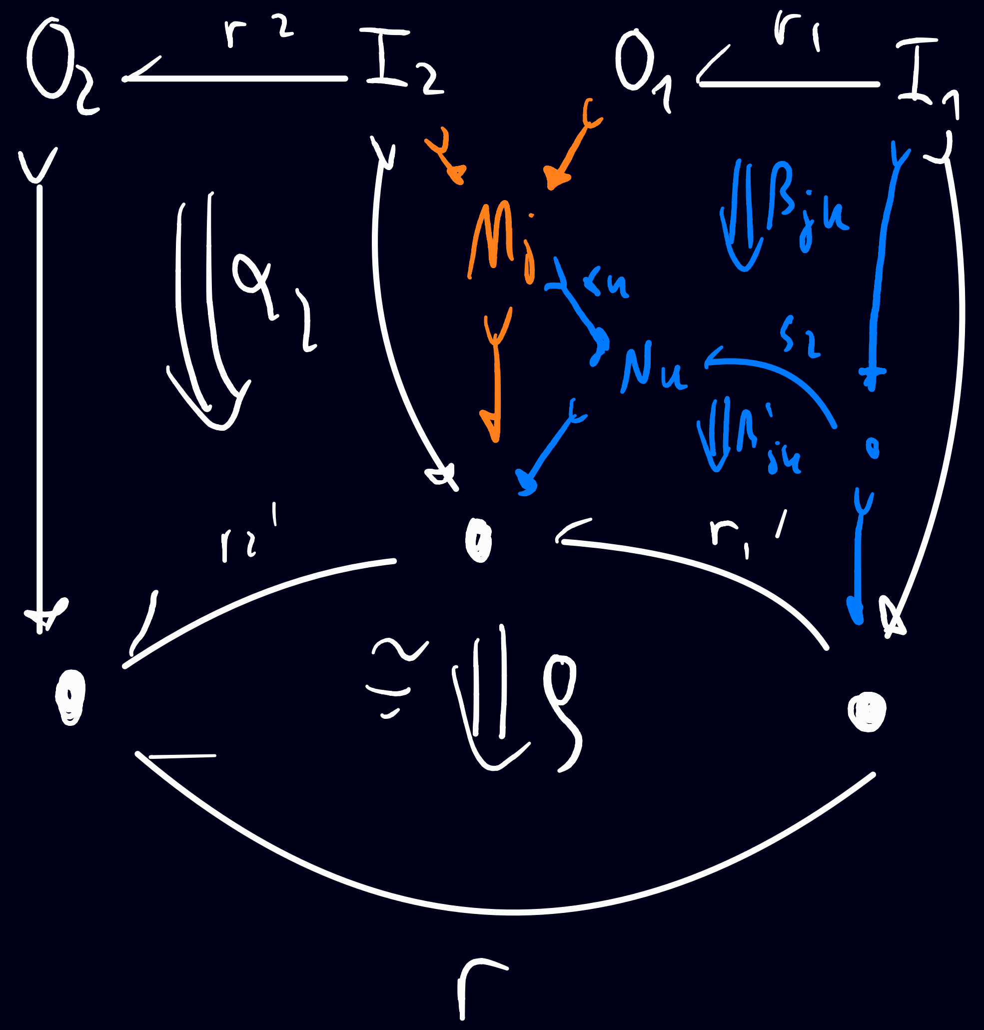
$\forall$



: $\exists!$

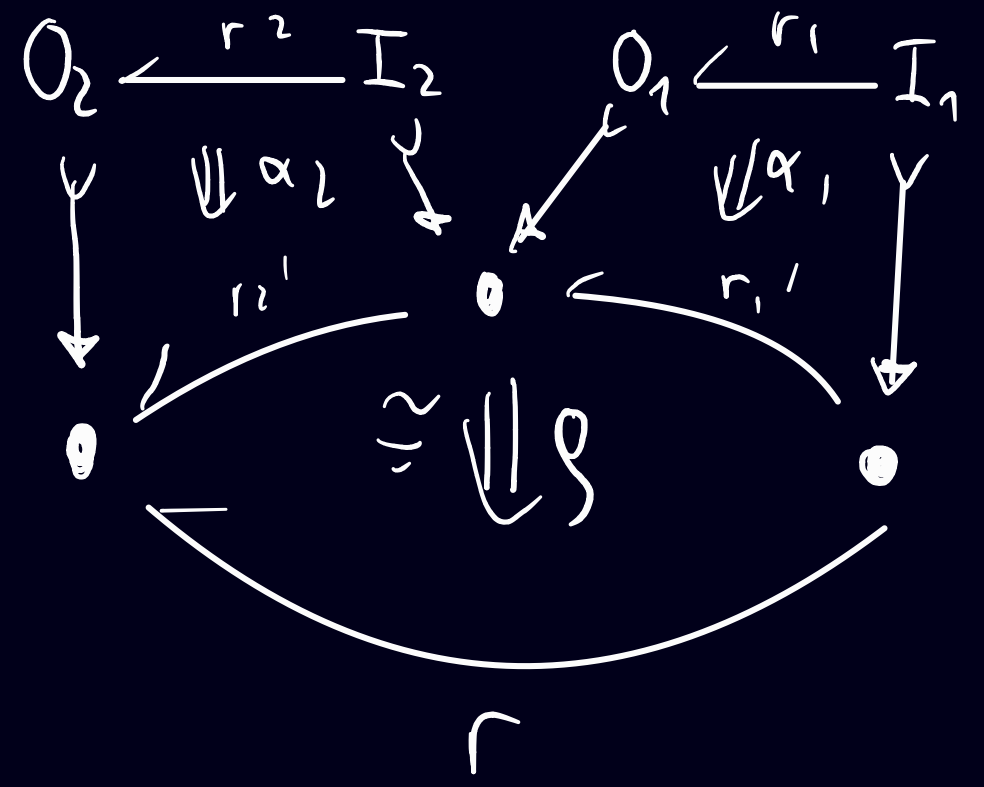


: $\exists!$

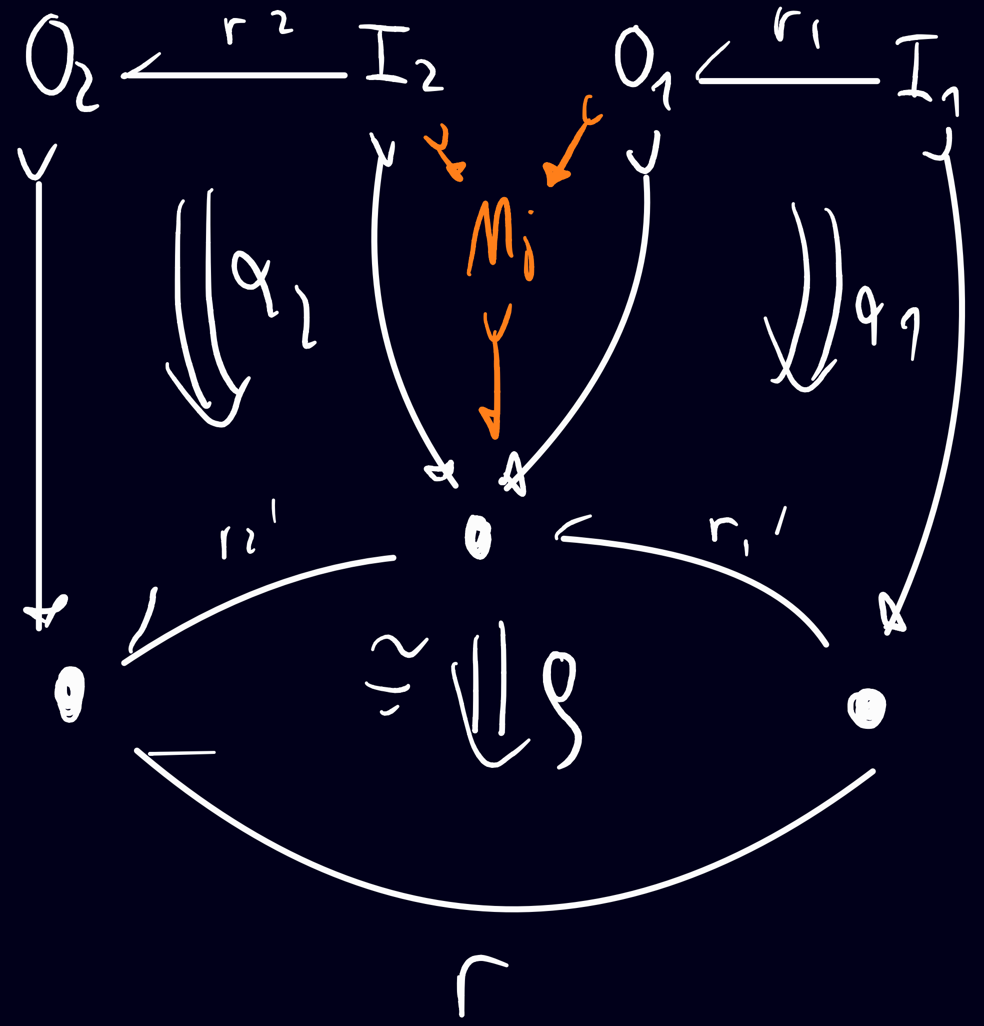


21 Proof (sketch): Assuming chosen cleavages:

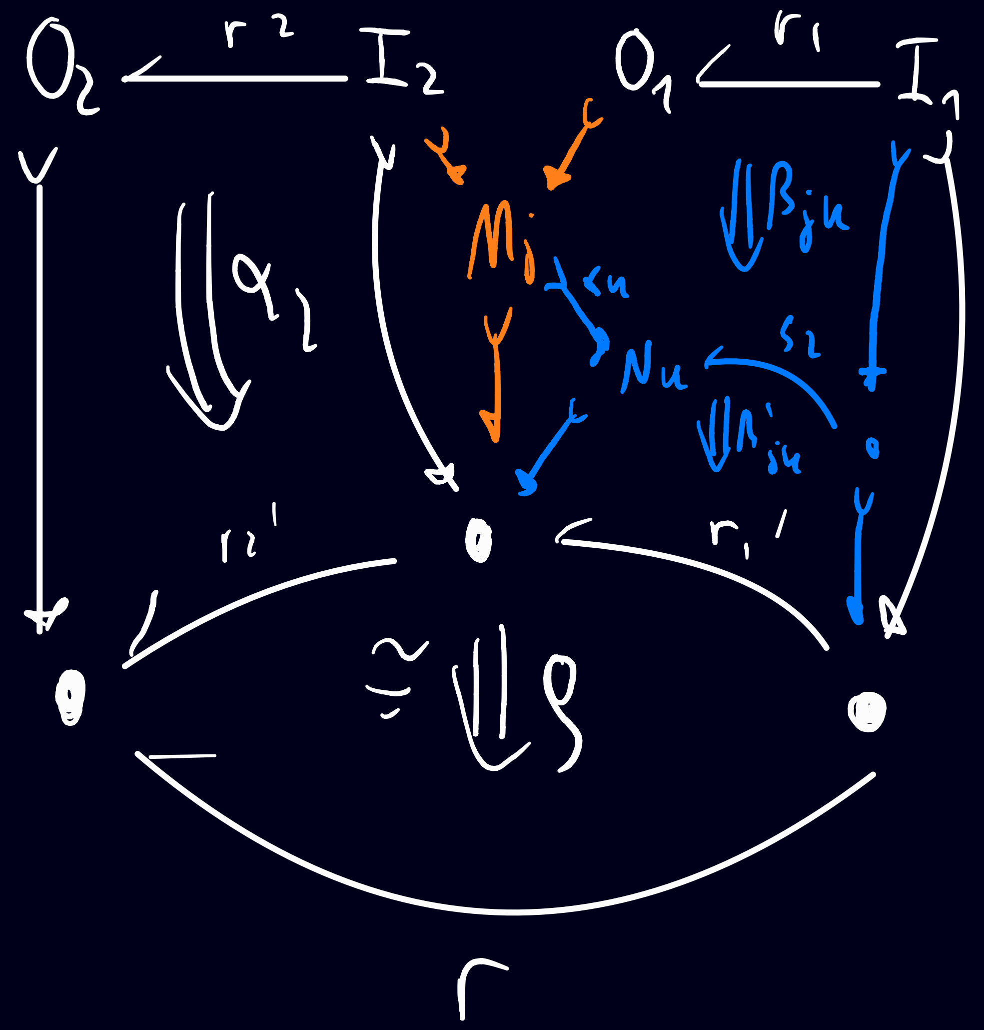
$\forall$



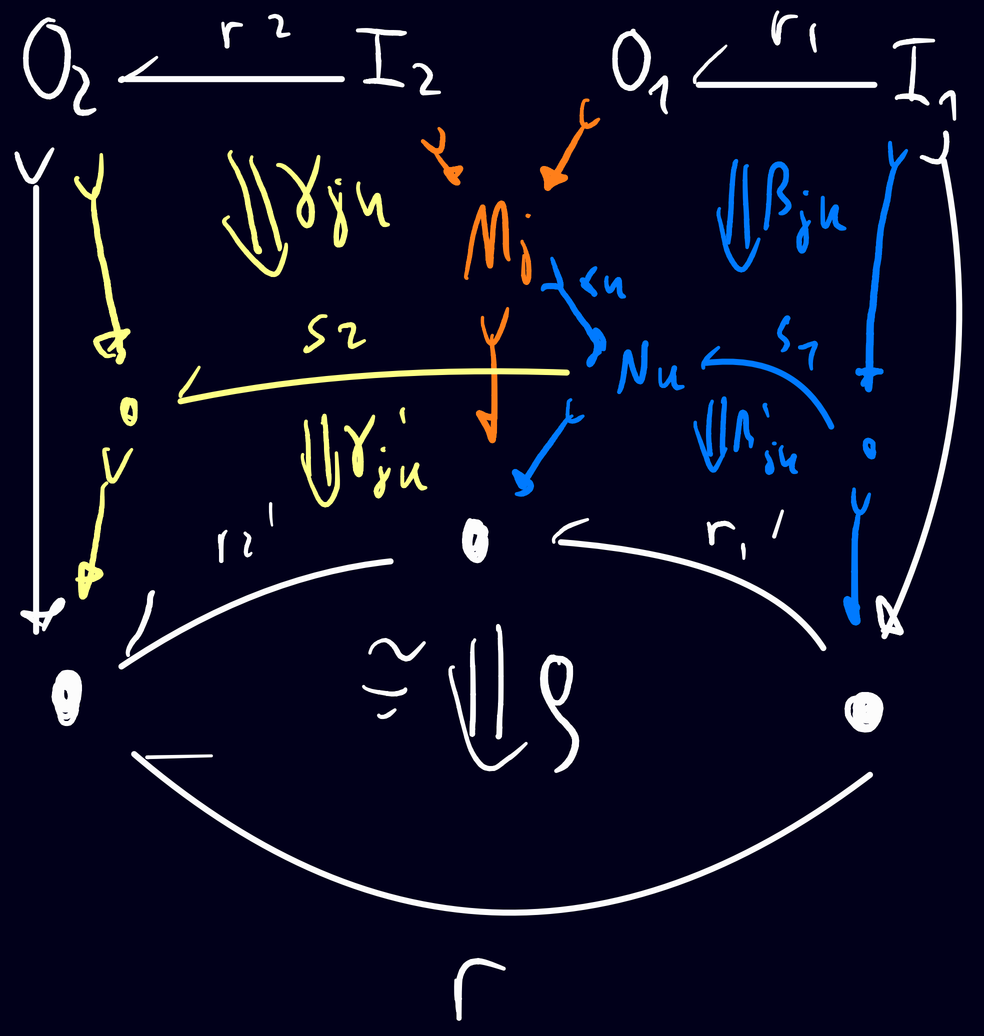
: $\exists!$



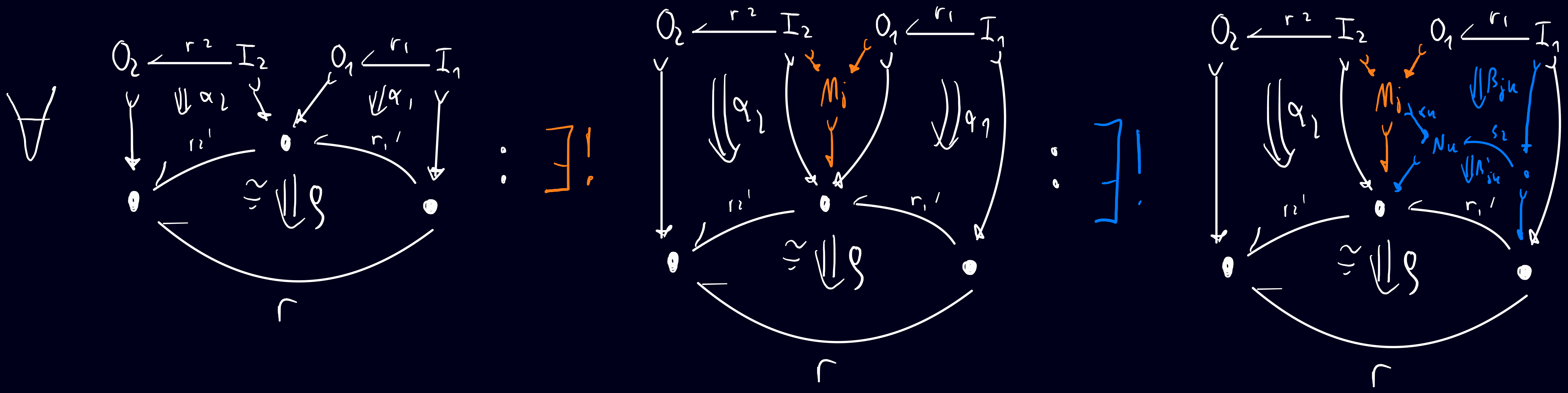
: $\exists!$

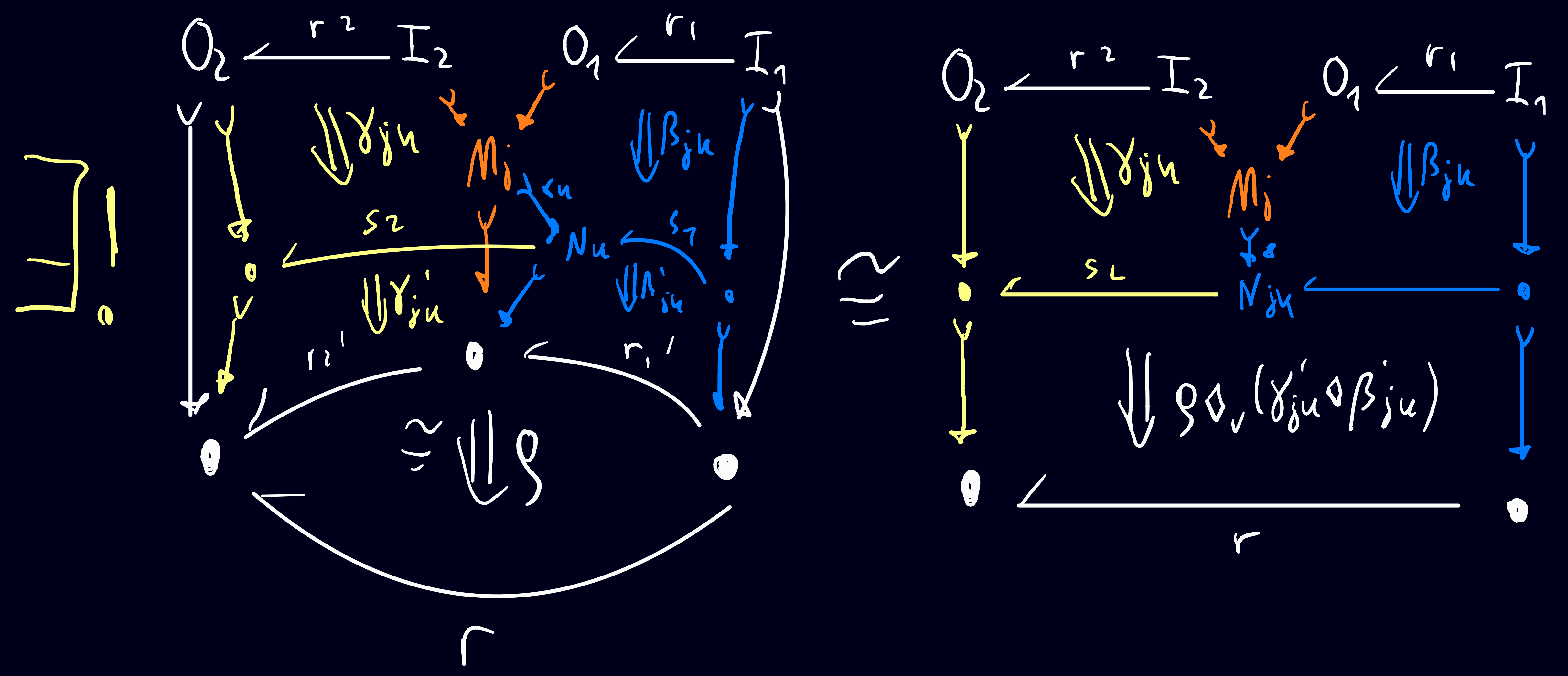


$\exists!$



21 PROOF (SKETCH): ASSUMING CHOSEN CLEAVAGES:



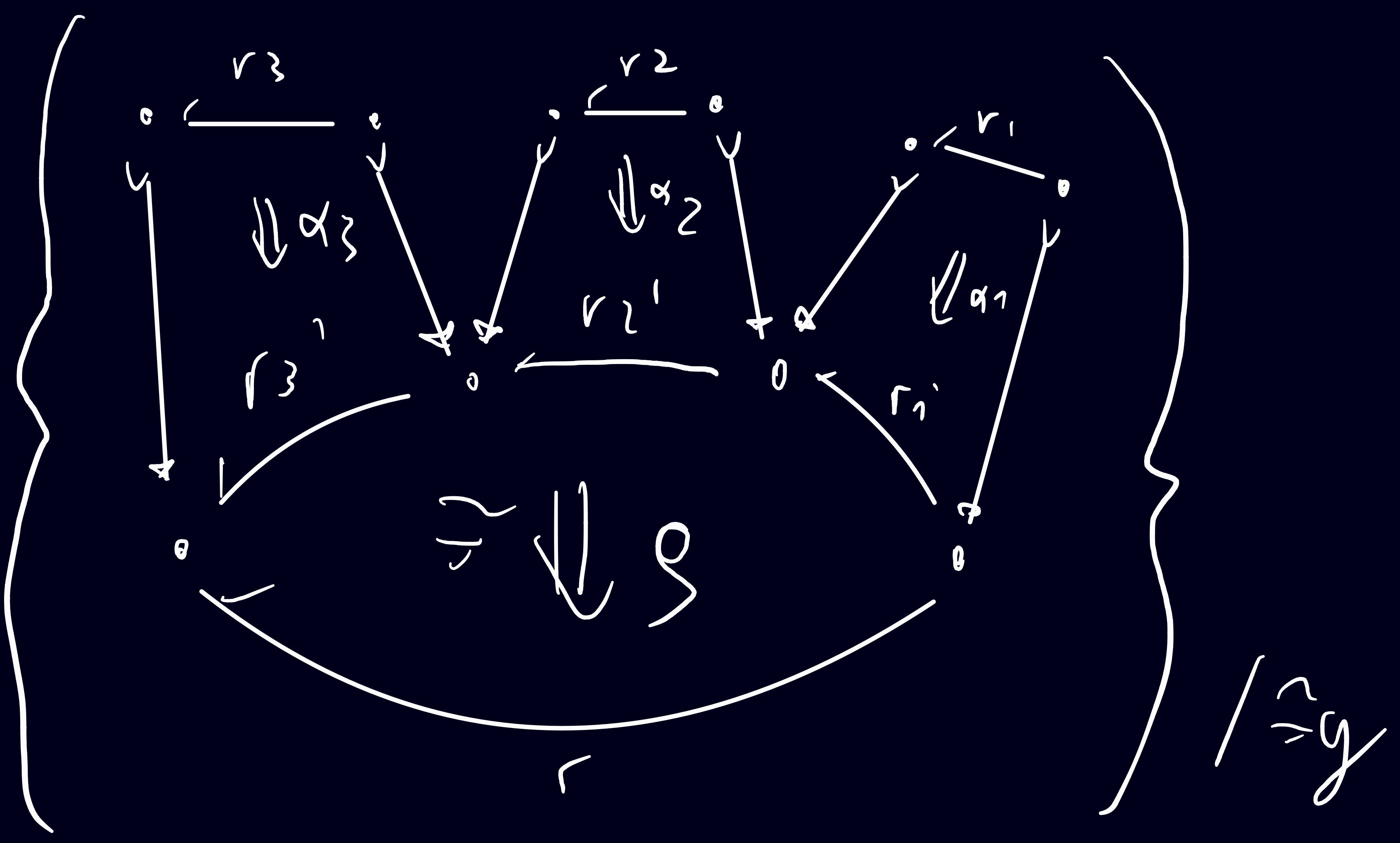


(21) Proof (sketch): Assuming chosen cleavages:

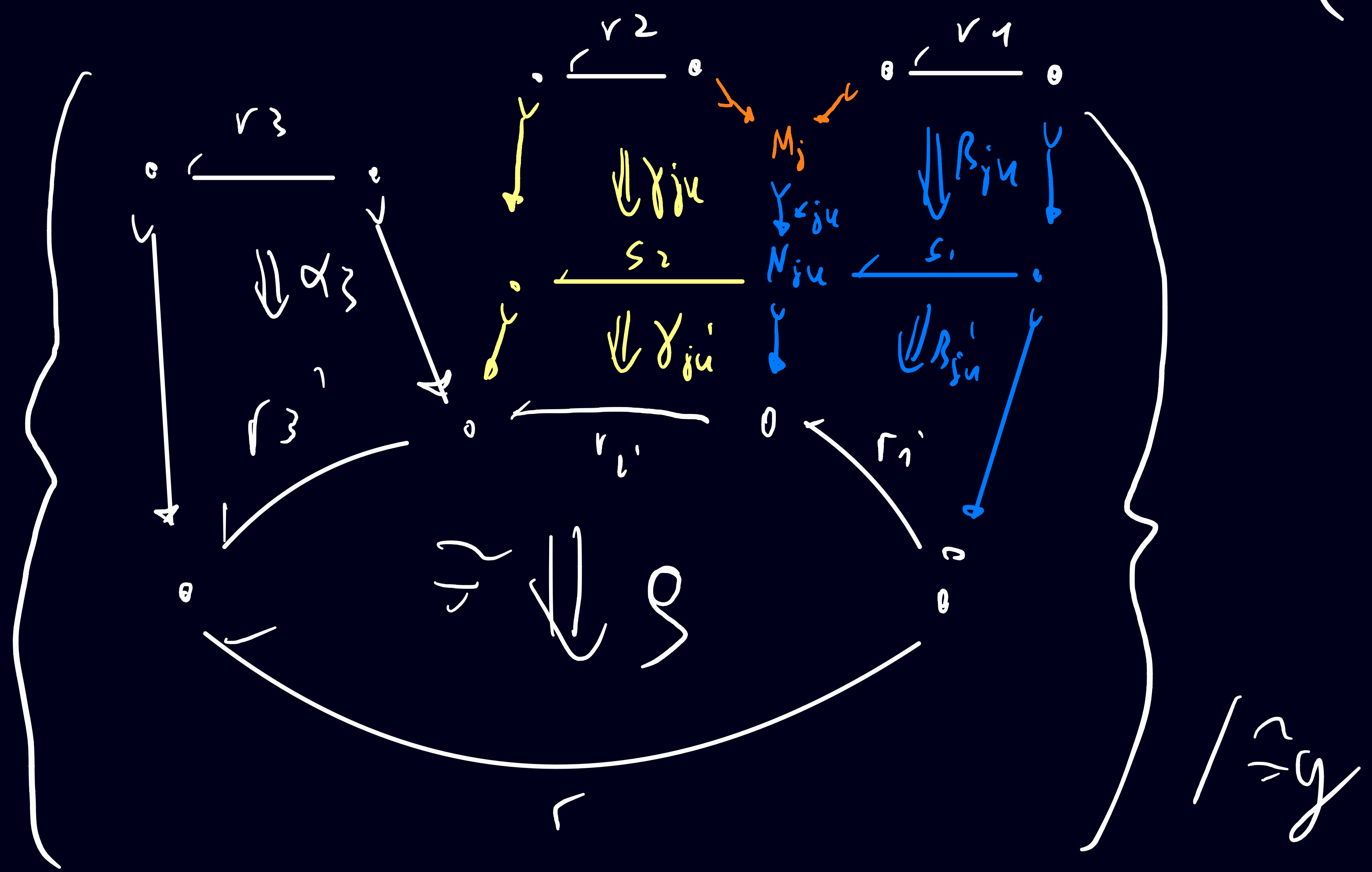
(22) CLAIM: $\hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r) \cong \hat{\Delta}_{(r_3 \odot r_2) \odot r_1}$

PROOF (SKETCH):

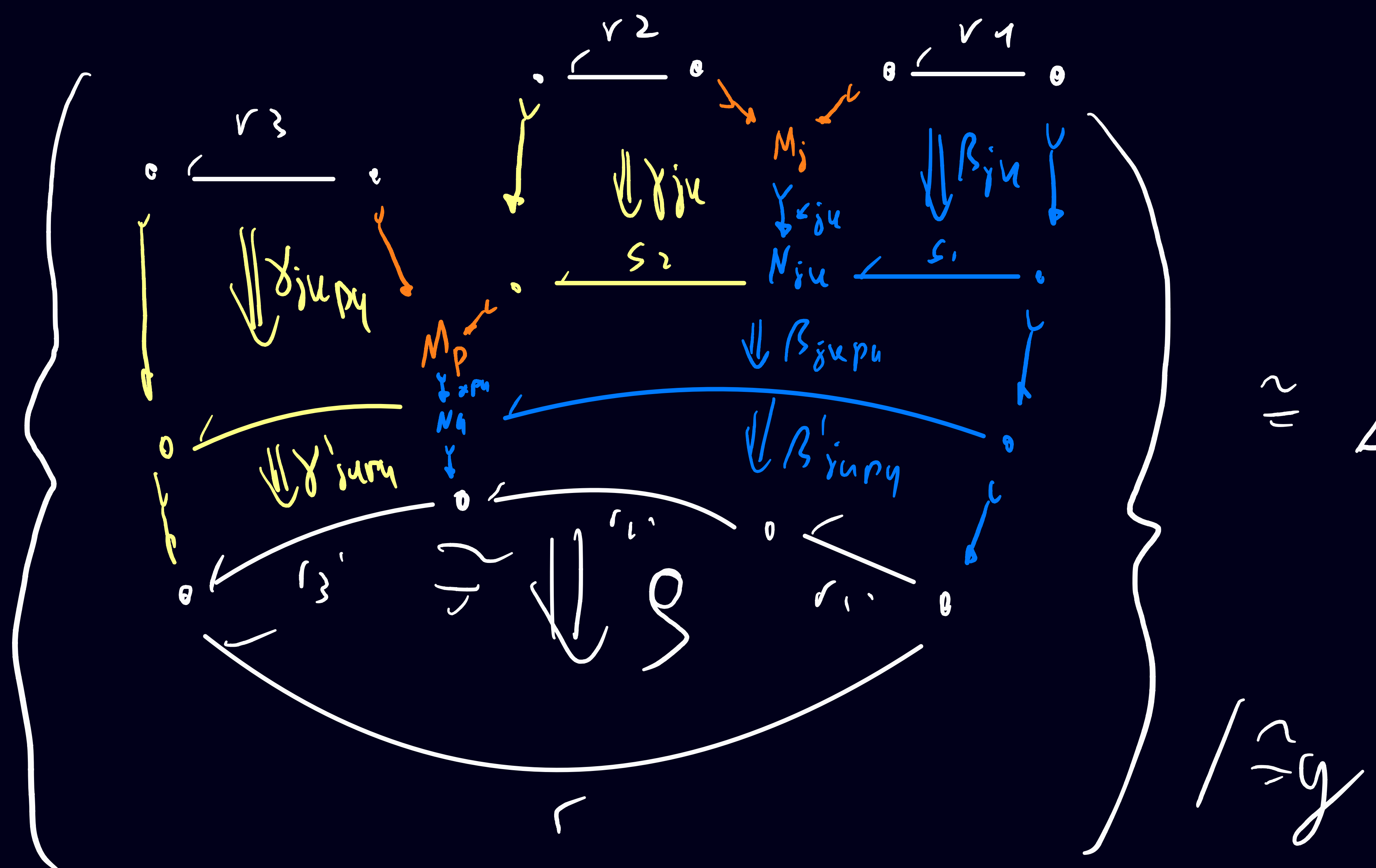
$$(\hat{\Delta}_{r_3} * \hat{\Delta}_{r_2} * \hat{\Delta}_{r_1})(r) \cong$$



$\cong$



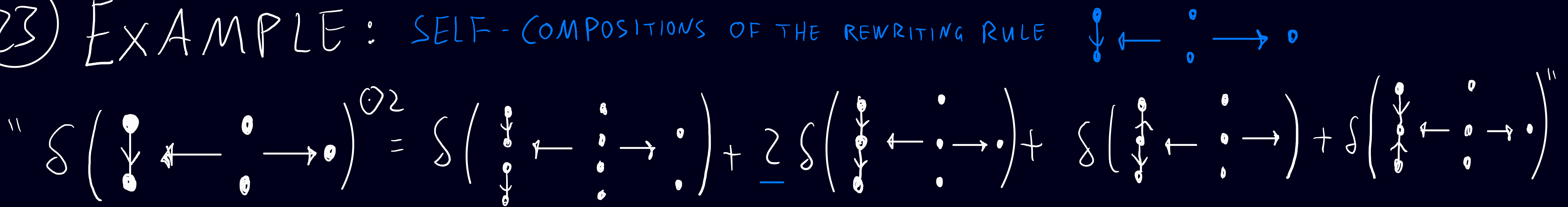
$\cong$



$$\cong \hat{\Delta}_{r_3 \odot (r_2 \odot r_1)}(r)$$

$\square$

㉓ EXAMPLE: SELF-COMPOSITIONS OF THE REWRITING RULE

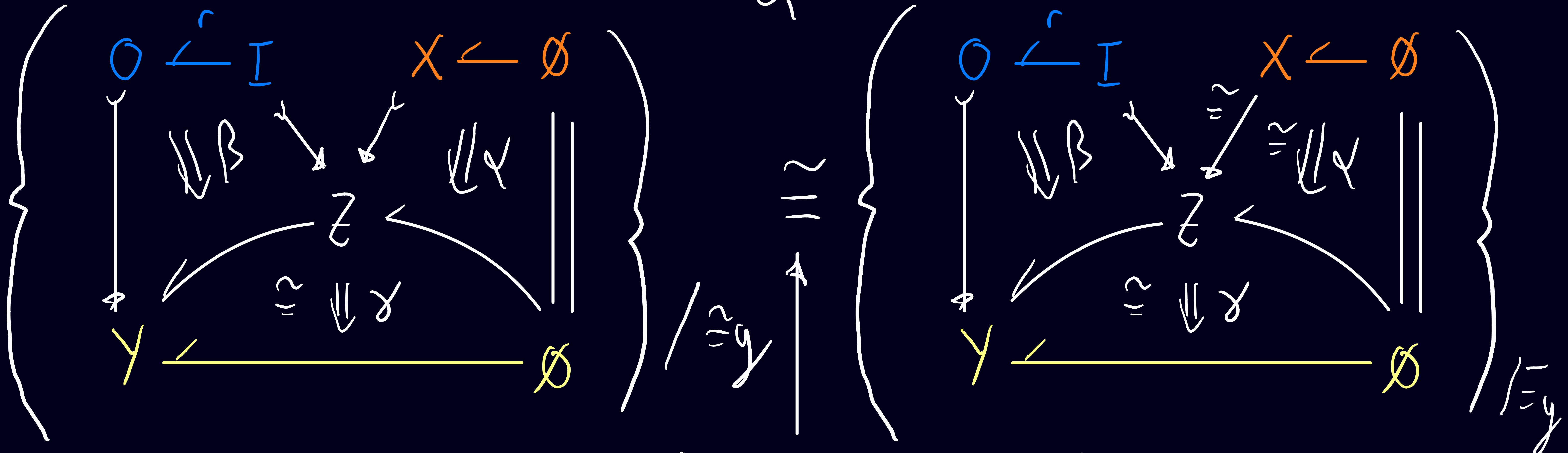


5 CONTRIBUTIONS TO $\hat{A}_{r_2 \circ r_1}$:

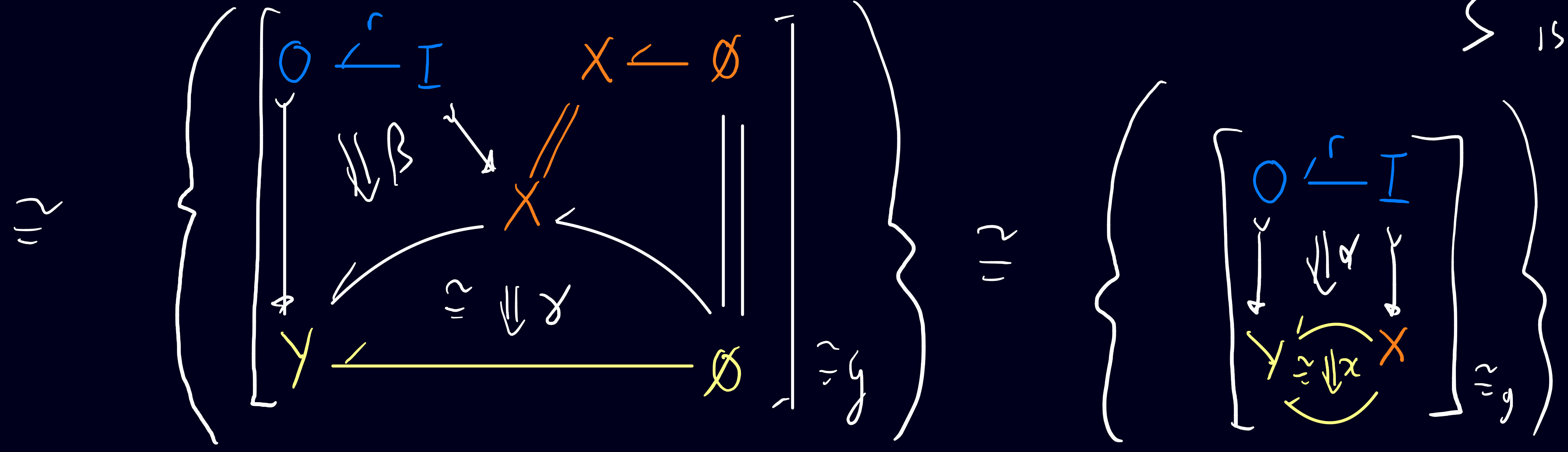
(24) Counting rewriting sequences

- "$\rho(\mathbb{S}(r))|X\rangle = \rho(\mathbb{S}(r))\rho(\mathbb{S}(X \leftarrow \emptyset))|\emptyset\rangle = \sum_{\alpha} \rho(\mathbb{S}(r_\alpha(X) \leftarrow \emptyset))|\emptyset\rangle$"

$(\hat{\Delta}_r * \hat{\Delta}_{X\leftarrow\emptyset})(Y\leftarrow\emptyset) \cong$



S is a MOPF ⇒ "lifts" isos!

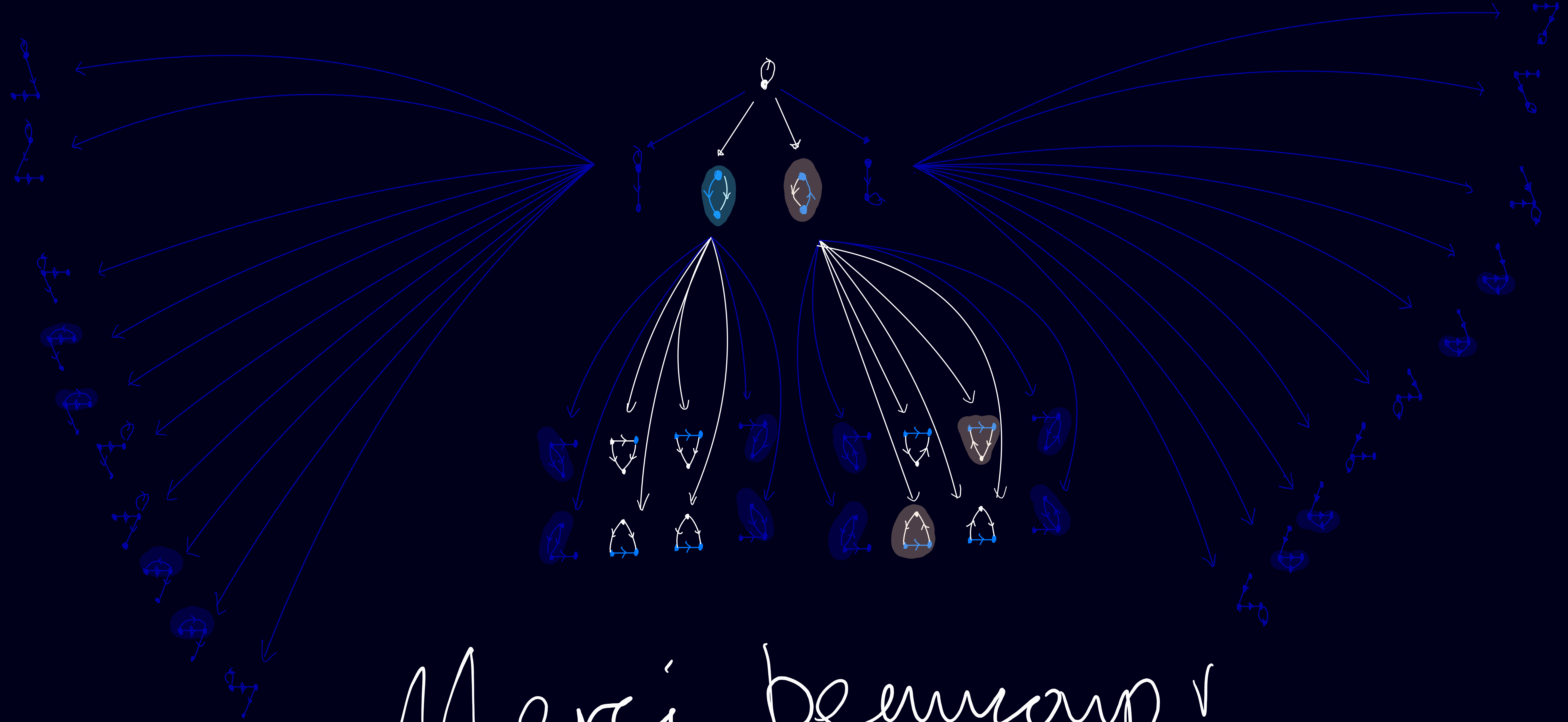


(25) "# OF WAYS TO REWRITE $\color{orange}X$ VIA APPLYING RULE $\color{blue}r$":

$$\int^{Y\in\mathbb{D}_0}\left(\hat{\Delta}_{r} * \hat{\Delta}_{X\leftarrow\emptyset}\right)(Y\leftarrow\emptyset) \cong \coprod_{\alpha\in\mathbb{D}_1}\left\{\begin{tikzcd} O \arrow[d, hook] & I \arrow[l, "r"'] \arrow[d, hook] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd}\right\}\Big/\sim_Y$$

WHERE:

$$\forall \begin{tikzcd} Y \arrow[d, "\cong"'] & \\ Y' & X \arrow[ul, "s"'] \arrow[l, "s'"] \arrow[ul, phantom, "\Downarrow\gamma"] \end{tikzcd} : \quad \begin{tikzcd} O \arrow[d, hook] & I \arrow[l, "r"'] \arrow[d, hook] \arrow[dl, phantom, "\Downarrow\alpha"] \\ Y & X \arrow[l, "s"] \end{tikzcd} \sim_Y \begin{tikzcd} O \arrow[d, hook] & I \arrow[l, "r"'] \arrow[d, hook] \arrow[dl, phantom, "\Downarrow\gamma\circ\alpha"] \\ Y' & X \arrow[l, "s'"] \end{tikzcd}$$



Merci beaucoup!